REPORT RESUMES

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THIS TEXT ON ANALYTIC GEOMETRY AND CALCULUS IS A CORRESPONDENCE COURSE DESIGNED FOR ADVANCED PLACEMENT OF HIGH SCHOOL STUDENTS IN COLLEGE. EACH OF THE 21 LESSONS INCLUDES READING ASSIGNMENTS AND LISTS OF PROBLEMS TO BE WORKED. IN ADDITION, SUPPLEMENTARY EXPLANATIONS AND COMMENTS ARE INCLUDED THAT (1) PROVIDE ILLUSTRATIVE EXAMPLES OF CONCEPTS AND TECHNIQUES DISCUSSED IN THE TEXT, (2) CLARIFY IMPORTANT DEFINITIONS AND PROOFS GIVEN IN THE TEXT, AND (3) BROADEN THE SCOPE OF THE COURSE BY INTRODUCING IMPORTANT CONCEPTS NOT DETAILED BY THE TEXT. ANOTHER REPORT ON THIS PROJECT IS ED 010 392. (GC)

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Correspondence STUDY

COURSE

AN ADVANCED PLACEMENT COURSE
ANALYTIC GEOMETRY AND CALCULUS
Mathematics XVxAP

THE UNIVERSITY OF NEBRASKA
EXTENSION DIVISION
LINCOLN, NEBRASKA 68508

SUPERVISED CORRESPONDENCE STUDY SERIES

AN ADVANCED PLACEMENT COURSE IN ANALYTIC GEOMETRY AND CALCULUS
(Mathematics XVxAP)

by

Walter E. Mientka

John J. DeRolf, Editor

Material for the Student

UNIVERSITY EXTENSION DIVISION
The University of Nebraska
Lincoln, Nebraska

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The author has contributed articles to the following learned journals: *Scripta Mathematica*, *Proceedings of the American Mathematical Society*, *The Indian Mathematics Student*, *Journal of Research of the National Bureau of Standards*, *The Superior Student*, *Applied Scientific Research*, *Journal of the Indian Mathematical Society*, and numerous articles under a research program supported by the United States Air Force.

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DIRECTIONS TO THE STUDENT

This mimeographed booklet called the Material for the Student is your study guide for the course. It is sometimes referred to as the syllabus.

As you proceed through this course, you will find it easy to follow directions, for at each step you will be told exactly what to do next--what materials to study, what written work or activities are to be done, when to take each test, what papers to mail, etc. Here are a few preliminary explanations that will help you understand how to study by correspondence.

As you study this course, you will frequently run across five asterisks (* * * * *). These are to be your stop signals. Whenever you come to them, be sure to complete the work which has been assigned before you study further.

Requirements

Courses are generally arranged in units. They may not be of equal length; you will be told at the beginning of each unit about how much time you may expect to spend on it. If you wish, you may take more time than is indicated, or less time; but to finish the course in a semester of eighteen weeks it will be necessary to follow the schedule quite closely.

It is expected that you will spend approximately sixty to eighty minutes each day on this course--about the same amount of time as high school students generally spend on each of their courses.

Your Supervisor and Your Teacher

Your supervisor is the person in your own community who has been appointed by your superintendent or high school principal to assist you. He will give you the materials furnished by our department and make any special arrangements that may be required. He will also administer all tests and be responsible for mailing your papers and other materials to the Correspondence Center.

Your teacher is the person at the Correspondence Center who has charge of your work. To avoid confusion let us call him your correspondence instructor. He will take a very personal interest in your progress, and you will soon feel that you know him well. He is eager to help you and to make any special arrangements that may be needed to give you the greatest value from this course.

How to Get Help from Your Instructor

Feel free to call on your correspondence instructor for help and advice. Always consult your supervisor when you do so; he may be able to clear up your difficulty. When you write a letter to your instructor or enclose a note with your work, let your supervisor see what you have written and give it to him for mailing. He can be of most help to you if he knows and approves everything that is done. When you, your supervisor, and your correspondence instructor work in close cooperation, this course can be made of greatest value.

Written Assignments

When you are asked to send in written material (except when paper is provided in the course), we prefer that you write on $8\frac{1}{2}$ x 11 inch paper. All material must be legibly written. (Pencil work will be acceptable if done neatly with a pencil that writes black.) It is recommended that you write on both sides of the paper, unless otherwise directed. However, we do not wish to have the writing crowded.

Test Instructions

Explicit directions, such as when to take tests and what written work to send in, will be found in the units at the time when such work directions are to be followed. Two types of tests appear in the course. The Self-Check Tests are planned to help you know how well you have mastered your assignments. They are taken without supervision and are not to be mailed to the Correspondence Center. The tests over the units will be administered by your supervisor, who will mail them to the Correspondence Center.

After completing a supervised test, continue to work ahead unless you receive instructions to do otherwise. If the correspondence instructor decides, after checking your test, that you have not sufficiently mastered the work covered by the test, your supervisor will be notified promptly. He will direct you to do further work in accordance with the suggestions of the correspondence instructor.

Permanent Place for Material

In working through this course, you may find it more convenient to remove the binding and place the sheets in a loose-leaf notebook.

Your supervisor will provide a safe place in which to keep all the papers that are returned from the Correspondence Center. When each lesson or paper is corrected, it will be returned to him. He will, in turn, allow you to have the corrected paper for a reasonable length of time in order that you may benefit from the corrections and suggestions made by the correspondence instructor. At any time, you may have access to your papers by securing permission from your supervisor. Be sure to study the teacher's comments and suggestions on your papers and then return them to your supervisor.

The mailing directions are given on the colored sheet. Fill in the questionnaire which follows the mailing directions.

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MAILING DIRECTIONS

Whenever material is to be sent to the Correspondence Center, follow these mailing directions.

Ask your supervisor for an endorsement wrapper and fill it in as follows:

Name ....(print your name)
Course ....(give title and number)
Unit No. ....(fill in the number of the unit)
Test ....(fill in the number and letter)
Papers ....(what is included)
Date Mailed ....(date material is mailed)

On the lines provided, write plainly the name and address of your supervisor. Fold the material in the wrapper, following the directions on the wrapper. Give your work to your supervisor, who will mail it to the Correspondence Center.

* * * * *

QUESTIONNAIRE

Please fill out this questionnaire and mail it with the first mailing in the course. There are many things we should like to know about you as we read your papers.

Name Age Sex

School Address Grade Birthday

Home Address No. in your class

Name of Supervisor

Do you live in town or country?

What are the names of your parents or guardian?

What is your father's occupation?

Does your mother have an occupation outside your home?

What are the hobbies of your parents?

How many brothers and sisters do you have?

What work do you do outside of school?

When do you expect to graduate from high school?

What do you expect to do upon completing high school?

What is your favorite subject?

What subject do you like least?

What high school mathematics courses have you had?

Why are you taking this very demanding mathematics course?

Will you study on this course at home or at school?

Do you take part in (1) school plays?

(2) any musical groups (3) any other group?

What are your hobbies?

Do you enjoy reading? What type of reading do you enjoy most--
books, magazines, newspapers?

What papers or magazines do you read?

Which articles or sections of the papers and magazines interest you most?

Do you have access to a library? Where?

Do you make use of a library?

Have you worked on any special projects where you have used mathematics extensively?

What are your favorite radio or television programs?

What traveling have you done?

What academic honors have you had?

If you wish to add other comments about yourself which will help us know you better, please feel free to do so. If you have a picture of yourself, please send it along.

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THE UNIVERSITY OF NEBRASKA
UNIVERSITY EXTENSION DIVISION
HOME STUDY COURSE

PREFACE

Mathematics is a subject which requires more than a minimal amount of effort on behalf of the student if he is to succeed in mastering its fundamentals. In particular the study of the calculus will reap many harvests if one is willing to work and work consistently.

This course, based on the text Elements of Calculus and Analytic Geometry by George B. Thomas, Jr., is the first of a sequence of two courses. The material to be covered in this course is essentially that which is found in the first five chapters of this text. Another text is being included as a supplementary reference for this course. This is Edward A. Cameron's Algebra and Trigonometry. This reference is included to insure that the student has a good text to use as a review source for the first few lessons and as a practical reference thereafter.

The subject is built up gradually and one must understand each step before proceeding to the next. The chapters to be covered are not disjointed, they depend upon one another. Consequently, one cannot know one section well and others not so well and expect to succeed. A conscientious disciplined schedule of study is absolutely necessary for an average or above average performance.

Before beginning your first lesson you should read carefully the preface found on pages v-vi of the text, as well as the general instructions for correspondence study on the colored sheet at the front of this syllabus.

This course is divided into 21 lessons. Each lesson includes a reading assignment in the text and a list of problems to be worked. In almost every instance further supplementary explanation and comments will be found in this syllabus. The supplementary explanations should be regarded as a part of the reading assignment. This section is included with the following purposes in mind:

1. To provide a wide variety of examples illustrating the concepts and techniques which are discussed in the text

2. To supplement and clarify important definitions and proofs given in the text

3. To broaden the scope of the course by introducing and discussing important related concepts not treated in the text.

Please read and study the assigned section(s) before you read the supplementary explanation and before you attempt your assignment. You should build up a list of the definitions and the important theorems as you progress in the course. Additions to this list should be made after each lesson. Moreover you are strongly urged to read the text with paper and pencil at hand, so that you will be able to follow the proof of a theorem by writing out each step, filling in additional details where necessary.

Feel free to send in any questions you may have along with your assignment, since it is essential that you master each topic in the text before going on to the next.

Textbook

George B. Thomas Jr. Elements of Calculus and Analytic Geometry. Reading, Massachusetts: Addison-Wesley Publishing Company, Inc., 1959 (Second Printing 1962).

Edward A. Cameron. Algebra and Trigonometry. New York: Holt, Rinehart and Winston, Inc., 1960.

* * * * *

Lesson 1

REVIEW OF ALGEBRA AND TRIGONOMETRY

Reference Material

Any algebra and trigonometry texts which might be available can be used. For a specific reference the text Algebra and Trigonometry by Edward A. Cameron. New York: Holt, Rinehart, and Winston, Inc., 1960, is suggested.

Supplementary Explanation

1. Review extensively your algebra and trigonometry. The following topics should be included in your review: fractions, factoring, inequalities, absolute value, linear equations, quadratic equations, completing the square, quadratic formula, mathematical induction, binomial theorem, logarithms, theory of equations, exponents determinants, systems of equations, functions, graphs, progressions, definition of the trigonometric functions, radian measure, inverse trigonometric functions, graphing of the trigonometric functions and their inverses, fundamental identities, trigonometric equations, DeMoivre's Theorem, law of sines, law of cosines, and triangle solving.

2. Notice that beginning on page 565 in the Thomas text is listed most of the formulas from elementary mathematics which will be needed for this course.

3. After you have spent as much time as you think necessary on your review then, before proceeding with the course, take the algebra and trigonometry examination found on the following pages. A maximum of two hours may be used for this examination and when finished, check your answers with those found at the end of this lesson. If your score is at least 75 per cent, then proceed with the course. If not then repeat 1, 2 and 3 above.

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Be honest with yourself. This lesson with a diagnostic examination allows you to seek out and rectify weak areas in your preparation. To proceed with the course before you have been able to complete the test with 75 per cent success would be foolish. Give yourself a chance; follow directions.

* * * * *

Algebra and Trigonometry Examination (two hours)

For numbers 1-21 inclusive, encircle the number indicating the proper answer. If none of (1), (2), (3), or (4) is the correct answer, (5) is then the proper answer.

1. By combining terms, the expression $[x(x^{\frac{3}{2}} \cdot x^{-\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}$ may be written

   (1) $x^{\frac{3}{2}}$
   (3) $x^{\frac{1}{2}}$
   (5) None of these

   (2) x^3
   (4) $x^{\frac{3}{4}}$

2. By combining terms, the expression $[x^{a-1} \cdot x^{a+1} \cdot x^{-a}]^{\frac{1}{a}}$ may be written

   (1) 1
   (3) $x^{\frac{a^2-a-1}{a}}$
   (5) None of these

   (2) x
   (4) x^a

3. The smallest value of k for which the equation

   $2x^2 + x - 6k = 0$ will have real roots is

   (1) 0
   (3) $-\frac{1}{50}$
   (5) None of these

   (2) 1
   (4) $-\frac{1}{48}$

4. In the equation $A = \frac{h}{2}(B + b)$, B is the unknown. The solving the equation for B is

   (1) $\frac{2A}{h} - b$
   (3) $\frac{2A - b}{h}$
   (5) None of these

   (2) $2A - h - b$
   (4) $\frac{Ah}{2} - b$

5. The cubic equation $ax^3 + bx^2 + cx + d = 0$ with a, b, c, d real will have at least one real root because

   (1) the cube root of d will be real

   (2) $b^2 - 4ac$ will be real

   (3) the graph of $y = ax^3 + bx^2 + cx + d$ is not a line

   (4) a cubic equation has only real roots

   (5) None of these

6. Let $f(x)$ be a function of x such that $f(x_1 + x_2) = f(x_1) + f(x_2)$ for all pairs of values x_1, x_2 of x, and in particular $f(1) = 2$. The value of $f(5)$ is

(1) 0	(3) 7	(5) None of these
(2) 5	(4) 1	

7. The values of x for which the inequality $5 > \frac{3}{x - 2}$ is true are

(1) $x > \frac{13}{5}$	(4) $x < \frac{13}{5}$
(2) $2 < x < \frac{13}{5}$	(5) None of these
(3) Any x for which either $x > \frac{13}{5}$ or $x < 2$	

8. By writing in terms of positive exponents and performing the indicated operations, one may write

$$\frac{(4x^2 + y^2)^{-1}}{4y^{-2} + x^{-2}}$$

as

(1) x^2y^2	(3) $\frac{x^2y^2}{(4x^2 + y^2)^2}$	(5) None of these
(2) $\frac{4y^2 + x^2}{4x^2 + y^2}$	(4) $\frac{4x^2y^2}{(x^2 + y^2)^2}$	

9. If $\log x = -3.1673$, the value of k for which

$$10^k < x < 10^{k+1}$$

is

(1) 3	(3) -3	(5) None of these
(2) -2	(4) 2	

10. The expression $\log\left(\frac{36}{35}\right) + \log\left(\frac{14}{5}\right) - 2\log\left(\frac{6}{5}\right)$ equals

(1) $\log 2$
(2) $\log \frac{10}{7}$
(3) 0
(4) 1
(5) None of these

11. The sum of the solutions of the equation $\sqrt{x + 10} + x = 2$

(1) -1
(2) 5
(3) -5
(4) 3
(5) None of these

12. The value of x which satisfies the equation $\frac{\log x}{\log x + \log 3} = 2$ is

(1) $\frac{1}{9}$
(2) 5
(3) 8
(4) -6
(5) None of these

13. The sum of the values of x which satisfy the equation

$$(2^x)^2 + 8 = 9(2^x)$$

is

(1) 3
(2) 0
(3) 1
(4) 5
(5) None of these

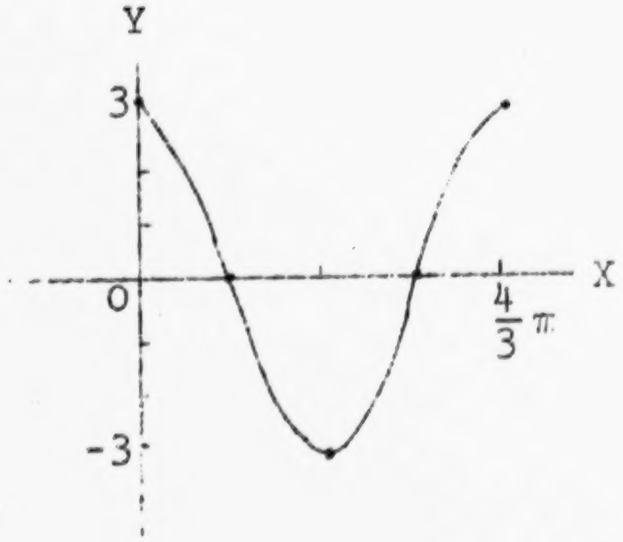
14. The expression $\frac{\cos 110^\circ - \tan(-320^\circ)}{2 \sin 170^\circ}$ written in a form in which only measures of angles between 0 and 90° appear is

(1) $\frac{\sin 20^\circ + \tan 50^\circ}{2 \sin 80^\circ}$
(2) $\frac{\cos 70^\circ + \tan 40^\circ}{2 \sin 10^\circ}$
(3) $\frac{-\cos 70^\circ - \tan 40^\circ}{2 \sin 10^\circ}$
(4) $\frac{-\sin 20^\circ - \tan 50^\circ}{2 \sin 80^\circ}$
(5) None of these

15. The values of k for which $\cos(\theta + \frac{k\pi}{2}) = \sin \theta \sin \frac{k\pi}{2}$ are

(1) the integers
(2) the even integers
(3) the odd integers
(4) only 1 and -1
(5) None of these

16. Part of the graph of $y = A \text{ cox } Bx$ looks as given in the sketch. The values of A and B should be



(1) $A = -3$, $B = \frac{2}{3}$ (4) $A = 6$, $B = \frac{3}{2}$

(2) $A = 3$, $B = \frac{3}{2}$ (5) None of these

(3) $A = 6$, $B = 3$

17. Tan $\frac{15\pi}{17}$ equals

(1) tan 16° (3) 0 (5) None of these

(2) tan $\frac{2\pi}{17}$ (4) - tan 16°

18. Sin(2 Arcsin $\frac{1}{3}$) equals

(1) $\frac{2}{3}$ (3) $\frac{4}{3}$ (5) None of these

(2) $\frac{4\sqrt{2}}{7}$ (4) $\frac{2\sqrt{2}}{3}$

$$\left[-\frac{\pi}{2} < \text{Arcsin } \frac{1}{3} = \text{Sin}^{-1} \frac{1}{3} < \frac{\pi}{2} \right]$$

19. The only values of x between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ that are solutions of the equation $\frac{\sqrt{3}}{2} \tan x - \sin x = 0$ are

(1) $0, \frac{\pi}{6}$ (3) $\pm\frac{\pi}{6}$ (5) None of these

(2) $0, \pm\frac{\pi}{6}$ (4) 0

20. On side BC of a right triangle in which C is the right angle, is a point D such that BD = 30, AC = 40 and $\sphericalangle$ DAC = 2 $\sphericalangle$ ABC. For finding the length of DC, the information given is

(1) sufficient

(2) contradictory

(3) inadequate because $\sphericalangle$ DAC or $\sphericalangle$ ABC must be given

(4) inadequate because length of AB or AC must be given

(5) None of these

21. The angles of a triangle ABC are such that angle B is $\frac{3}{2}$ of angle A and angle C is $\frac{7}{2}$ of angle A. The ratio of side b to side a is

(1) $\frac{3}{7}$

(2) $\frac{7}{3}$

(3) $\frac{2}{\sqrt{2}}$

(4) $\frac{3\sqrt{2}}{7}$

(5) None of these

Numbers 22-25: Work these problems on scratch paper.

22. Solve the pair of simultaneous equations

$$\log_{10}(x - 2y) = 1$$
$$5^{x+y} = 1$$

23. It is given that θ is any angle between 225° and 270°. What kind of number, positive or negative, will $(\sin\theta + \tan\theta)$ be? Give adequate reasons.

24. A cork sphere 1 foot in diameter sinks to a depth x in water. The depth is determined by the equation

$$2x^3 - 3x^2 + .24 = 0$$

State what you would expect to be true of the solutions of this equation.

25. An A.P. of three numbers is any three numbers of the form a, a + d, a + 2d.

A G.P. of three numbers is any three numbers of the form b, br, br^2.

Prove that no three distinct numbers form both an A.P. and a G.P.

* * * * *

Answers to the Algebra and Trigonometry Examination

1. (4)
2. (2)
3. (4)
4. (1)
5. (5)
6. (5)
7. (1)
8. (3)
9. (5)
10. (1)
11. (1)
12. (1)
13. (1)
14. (2)
15. (5)
16. (2)
17. (5)
18. (5)
19. (1)
20. (1)
21. (3)

22. $x = \frac{10}{3}$, $y = -\frac{10}{3}$

23. $(\sin\theta + \tan\theta)$ will be positive in the internal

$$225^\circ < \theta < 270^\circ$$

24. Would expect two positive roots and one negative

25. Assume that there exist three distinct numbers which form both an A.P. and a G.P., then $a = b$, $a + d = br$ and $a + 2d = br^2$. Thus we would have

$$r = \frac{a + d}{b}, \text{ or } r = \frac{b + d}{d}$$

and

$$r^2 = \frac{a + 2d}{b}, \text{ or } r^2 = \frac{b + 2d}{b}.$$

Now $\left(\frac{b + d}{b}\right)^2 = r^2$ and so it follows that

$$\frac{b^2 + 2bd + d^2}{2} = r^2.$$

Consequently we have $\frac{b^2 + 2bd + d^2}{b^2} = \frac{b + 2d}{b}$ or

$$1 + \frac{2d}{b} + \frac{d^2}{b^2} = 1 + \frac{2d}{b} \text{ or } \frac{d^2}{b^2} = 0,\ d = 0.$$

Hence all numbers would be the same. Thus it is impossible to have three distinct numbers which form both an A.P. and a G.P.

* * * * *

Written Assignment

Since this is a review assignment, no written lesson will be mailed to the correspondence instructor for evaluation.

* * * * *

Lesson 2

REVIEW OF THE NUMBER SYSTEM

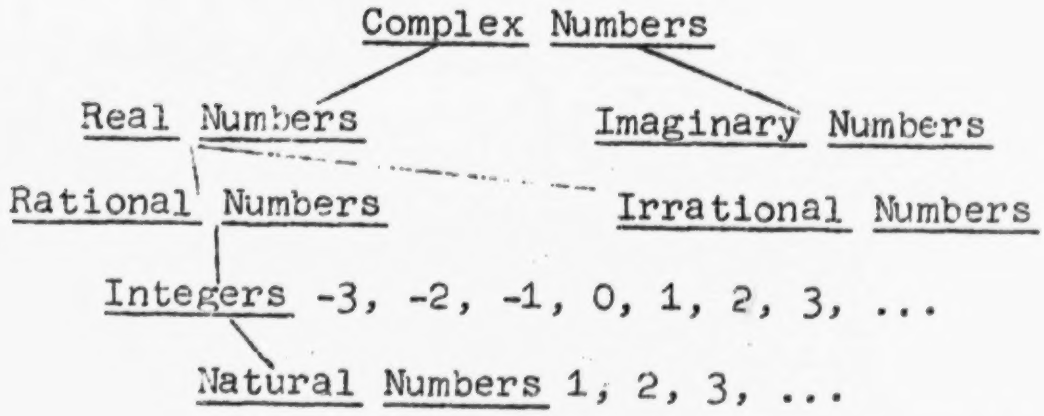
Reference Material

All material is found in the explanation in this syllabus.

Supplementary Explanation

1. So that there will be absolutely no confusion regarding numbers and the names and definitions associated with them the following schematic diagram is presented along with heuristic definitions.

Schematic Diagram



Heuristic Discussion

2. The natural numbers consist of the set of counting numbers 1, 2, 3, ... The integers consist of the set of positive and negative whole numbers and zero. The rational numbers are defined as numbers of the form $\frac{p}{q}$ where p and q are integers and q is never equal to zero (consequently expressions of the form $\frac{2}{0}$, $-\frac{7}{0}$ are not numbers since by definition $q \neq 0$). Note that all integers are rational numbers. (Why?) The irrational numbers are defined to be numbers which are not rational. For example $\sqrt{2}$, $\sqrt{3}$, and π are irrational numbers. They are generally characterized by the fact that they may be represented by an unending decimal which does not repeat. For instance $\pi = 3.14159 \cdots$ is an irrational number but $.333333 \cdots = \frac{1}{3}$ is a rational number.

Is .123123123 ⋯ a rational or irrational number? If rational write it as a ratio of two integers. The real numbers consist of the rational numbers and the irrational numbers. The imaginary numbers are numbers which involve $i = \sqrt{-1}$. The complex numbers are defined as numbers of the form $a + bi$ where a and b are real numbers and $i = \sqrt{-1}$.

3. Note that all numbers are complex numbers. For example 3 is a complex number since $3 = 3 + 0i$. Also, $\sqrt{2}\, i$ is a complex number since $\sqrt{2}\, i = 0 + \sqrt{2}\, i$. Observe that the definitions given above lead to the structure of our diagram. The numbers appearing above include those below in the chart.

Become familiar with the numbers considered above and the definitions. Be able to give several examples of each type. Three concluding remarks will be made concerning some of these numbers.

4. We will show the existence of (infinitely many) irrational numbers. Let N be any integer greater than 1 which is not a multiple of 10, and let $L = \log_{10} N$. Certainly L is a real number, so either L is rational or L is irrational. Suppose L is rational, say $L = \frac{p}{q}$, where p and q are positive integers with $q \neq 0$. Then $\log_{10} N = \frac{p}{q}$, so

$$N = 10^{\frac{p}{q}}.$$

Raising both sides of this equation to q^th power, we obtain

$$N^q = 10^p.$$

Now 10^p, being a positive integral power of 10, must end in 0. However N^q cannot end in 0. [Since N is not a multiple of 10, N must end in a non-zero digit. If N ends in 1, so does every positive integral power of N. If N ends in 2, the positive integral powers of N all end in 2, 4, 6, or 8. Similarly if N ends in 3, 4, 5, 6, 7, 8, or 9 we see that no positive integral power of N can end in 0.] This contradiction tells us that our assumption that L was rational must be false. Therefore L is irrational.

5. The set of all real numbers can be thought of as the set of all decimals (such as 0.25, 321.0, and 0.333 ...). As you can find in almost any algebra text, a real number is rational if and only if it can be expressed as a repeating decimal

(such as $0.142857\ 142857 \cdots = \frac{1}{7}$, or $0.625000 \cdots = \frac{5}{8}$), while a real number is irrational if and only if it can be expressed as a non-repeating decimal (such as $3.14159 \cdots = \pi$, or $0.47712 \cdots = \log_{10}3$).

Every real number having a decimal expansion which repeats in 9's also has a decimal expansion which repeats in 0's; for example,

$$0.4999 \ldots = 0.5000 \ldots$$

6. We will show that between any two distinct real numbers x and y there is both a rational number and an irrational number. Suppose that x is less than y and, for simplicity of notation, that x and y lie between 0 and 1. Then x and y have decimal expansions of the form

$$x = 0.A_1A_2A_3 \ldots$$

and

$$y = 0.B_1B_2B_3 \ldots,$$

where the A's and B's are digits. Moreover, by suggestion 5, we can assume that the given decimal expansion of x does not repeat in 9's.

Suppose that the given expansions of x and y differ for the first time in the k^{th} decimal place (i.e., $A_1 = B_1$, $A_2 = B_2$, $\ldots$, $A_{k-1} = B_{k-1}$, $A_k \neq B_k$). Then, since x is less than y, A_k must be less than B_k.

Also, since x does not repeat in 9's, at least one digit to the right of A_k, say A_m, must be less than 9. Now consider the real numbers

$$r = 0.A_1A_2 \ldots A_k \ldots A_{m-1}\ 9111 \ldots$$

and

$$i = 0.A_1A_2 \ldots A_k \ldots A_{m-1}\ 9010010001 \ldots$$

Both r and i coincide with x up to the $(m-1)^{st}$ place and are greater than x in the m^{th} place (since A_m is less than 9), so both r and i are greater than x. Moreover, both r and i coincide with y up to the $(k-1)^{st}$ place (since $A_1 = B_1$, $\ldots$, $A_{k-1} = B_{k-1}$) and are less than y in the k^{th} place (since A_k is less than B_k), so both r and i are less than y. Therefore r and i lie between x and y. But r repeats in 1's, so r is rational, while i is clearly non-repeating, so i is irrational.

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Written Assignment

Since this is a review assignment, no written assignment will be mailed to the correspondence instructor for evaluation.

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Lesson 3

DISTANCE FORMULAS

Reference Material

Thomas, Chapter 1, Sections 1, 2, and 3

Supplementary Explanation

1. Section 1 is essentially one which provides the student with a brief insight into the problems which can be solved using the calculus. Also a historical background of the subject is presented to the reader.

2. Section 2 should be a review section for you since it introduces once again the rectangular coordinate system.

3. Section 3 contains the notions of Directed Distance from the point A to the point B on a horizontal line and from the point C to the point D on a vertical line. Note the results very carefully. An immediate consequence of these notions is the distance formula d between the two points (not necessarily on a horizontal or vertical line) $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

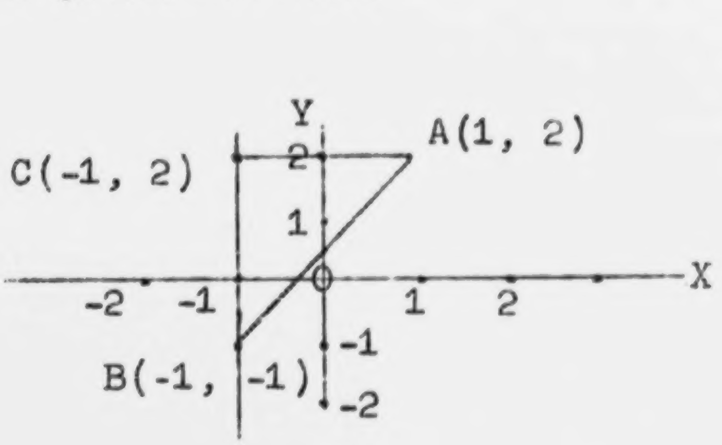
   $$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

   This formula will be extremely useful throughout the course.

4. Let me illustrate some of these ideas with an example.

   Problem 6, page 7:

   Solution: We are given the points A(1, 2) and B(-1, -1). Plotting these points we have



(a) C has coordinates (-1, 2)

(b) $\overline{AC} = \Delta x = -1 - (1) = -2$

(c) $\overline{CB} = \Delta y = -1 - 2 = -3$

(d) $AB = \sqrt{[1 - (-1)]^2 + [2 - (-1)]^2} = \sqrt{4 + 9} = \sqrt{13}$

5. Problem: Find an equation of a circle with center at (h, k) and radius r.

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Written Assignment

Remark to the student: Most of the answers to the problems which will be assigned are found in the back of the book beginning on page 517. You must present enough work for each problem to justify your answer.

Solve Problems 1-15 (odd-numbered problems), and Problems 16, 17, and 18, page 4 in the textbook.

Solve Problems 1-9 (odd-numbered problems), page 7 in the Thomas textbook.

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Give the prepared work for Lesson 3 to the supervisor who will complete the mailing procedure.

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Lesson 4

SLOPE AND EQUATIONS OF A STRAIGHT LINE

Reference Material

Thomas, Chapter 1, Sections 4 and 5

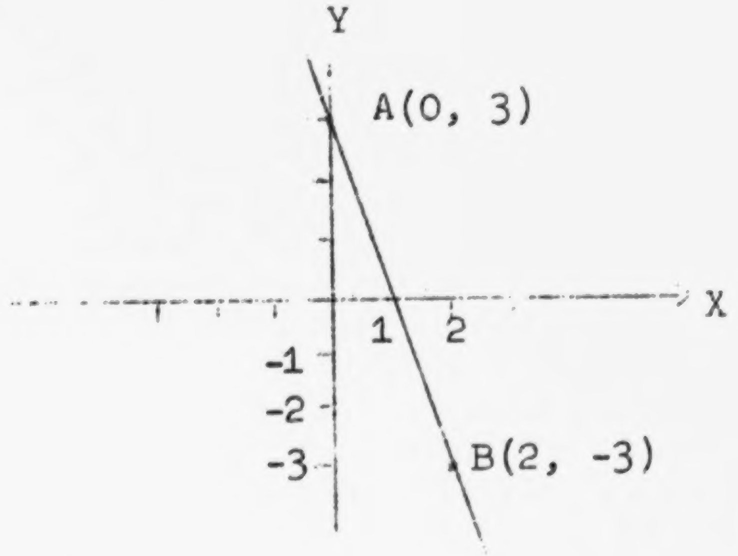
Supplementary Explanation

1. After you read and study Section 4, summarize the results either on scratch paper or in your own personal notebook. Your list should include the *different* ways in which the *slope of a line* is defined. Always keep in mind the condition for two lines to be parallel and the condition for two lines to be perpendicular. Note well the remarks (in smaller print) at the bottom of page 10 and at the top of page 11 regarding the meaning of the symbol ∞ and the fact that the arithmetical operations are not applicable to this symbol (it is not a number). Lastly, from remark 4 on page 10, we conclude that the slope of a horizontal line is zero and the slope of a vertical line is undefined.

2. Illustrative Examples:

Problem 8, page 12

Solution: Let m_{AB} represent the slope of the line AB and $m_{\perp AB}$ represent the slope of a line perpendicular to AB. Our figure is given by

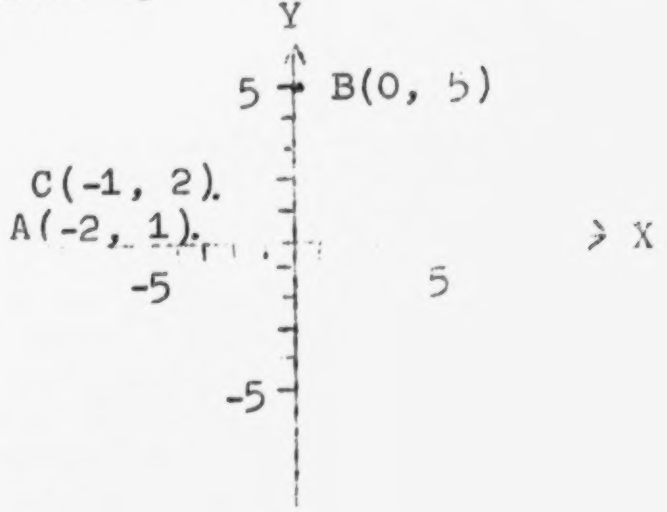


Now $m_{AB} = \frac{-3 - 3}{2 - 0} = \frac{-6}{2} = -3$ and since $m_{\perp AB} = -\frac{1}{m_{AB}}$, we have $m_{\perp AB} = -\frac{1}{-3} = \frac{1}{3}$.

Problem 24, page 12

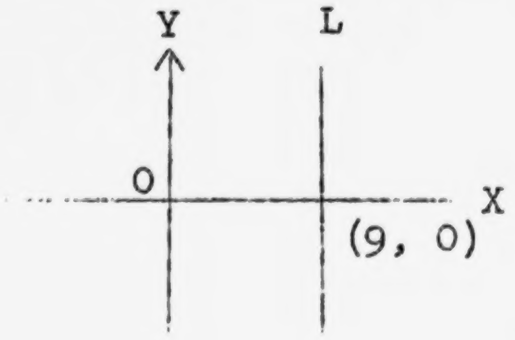
Solution: Let m_{AC} and m_{CB} represent the slopes of the lines AC and CB respectively.

Our figure is given by

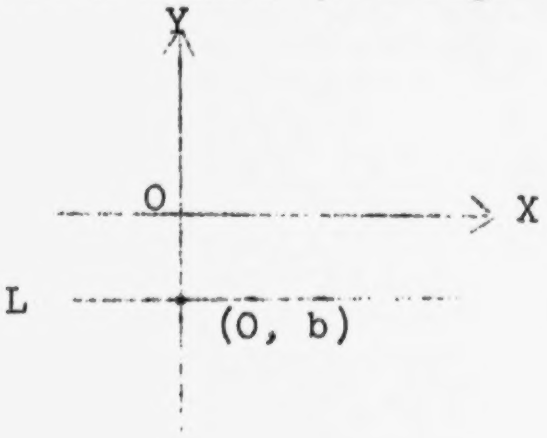


Now if the given points lie on a straight line then it must follow that $m_{AC} = m_{CB}$. Now $m_{AC} = \frac{2 - 1}{-1 - (-2)}$ $= \frac{1}{-1 + 2} = 1$ and $m_{CB} = \frac{5 - 2}{0 - (-1)} = 3$. Thus the given points do not lie on the same straight line.

a. A line L parallel to the x-axis has equation $x = a$, where a is any real number (see figure).



b. A line L parallel to the x-axis has equation $y = b$, where b is any real number (see figure).



c. Point-slope form: If we are given a point $P_1(x_1, y_1)$ and the slope m of a line L then the point-slope form of an equation for L is given by:

$$y - y_1 = m(x - x_1)$$

d. Slope-intercept form: From the point slope form we find that if the given point is (0, b) and the slope is m then we have $y - b = m(x - 0)$ or $y - b = mx$ and finally the slope-intercept form $y = mx + b$.

Note that the real number b represents the y-intercept i.e., the point where the line intersects the y-axis and m represents the slope.

e. The general equation of the straight line is given by $AX + BY + C = 0$, where A, B, and C are real numbers with at least one of A and B different from zero.

f. The intercept form: We shall derive this form which is not given in the text but which is sometimes very useful.

Let L be a line with x-intercept a and y-intercept b, where neither a nor b is 0. The L passes through the points (a, 0) and (0, b), so the slope of L is

$$\frac{b - 0}{0 - a} = -\frac{b}{a}.$$

Thus the slope-intercept form of the equation of L is

$$y = -\frac{b}{a}x + b.$$

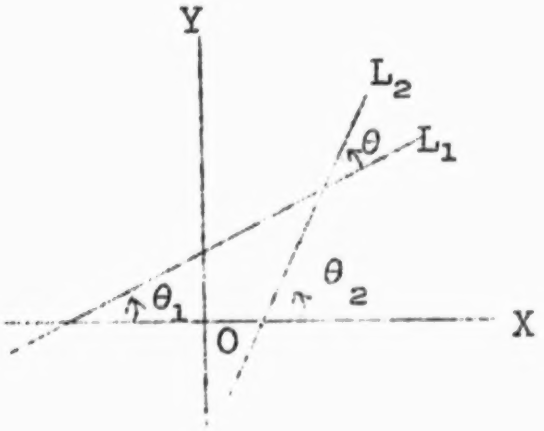
This equation can be written as bx + ay = ab or, dividing both sides by ab,

$$\frac{x}{a} + \frac{y}{b} = 1.$$

This equation is called the intercept form of the equation of L.

4. We now derive a result which will yield the angle between two lines.

Let L_1 and L_2 be two lines with respective slopes $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$. Let θ denote the angle from L_1 to L_2.



Then $\theta = \theta_2 - \theta_1$ so, by a fundamental trigonometric identity,

$$\tan\theta = \tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}.$$

Thus we obtain the relation

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}.$$

For example, suppose we wish to find the angle θ from the line AB to the line PQ, where A = (-2, -6), B = (8, 2), P = (2, 8) and Q = (4, -2). Then

$$m_1 = \text{slope of } AB = \frac{2 - (-6)}{8 - (-2)} = \frac{4}{5},$$

$$m_2 = \text{slope of } PQ = \frac{-2 - 8}{4 - 2} = -5$$

so

$$\tan\theta = \frac{-5 - \frac{4}{5}}{1 + (-5)\left(\frac{4}{5}\right)} = \frac{29}{15}.$$

θ can then be determined to the nearest degree, for example, from a table of tangents.

Note that if L_1 and L_2 are parallel (but not parallel to the y-axis) then $m_1 = m_2$ so (1) becomes

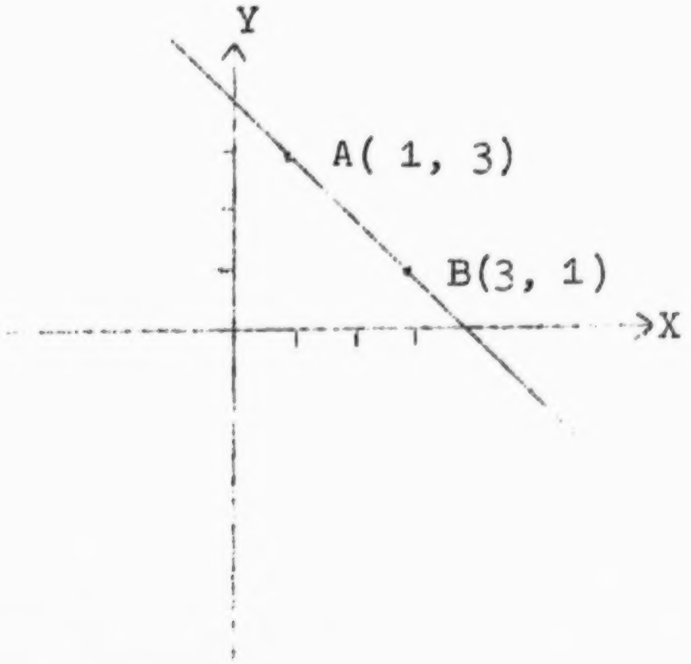
$$\tan\theta = \frac{m_1 - m_2}{1 + m_1^2} = 0$$

so $\theta = 0^\circ$. Also, if L_1 is perpendicular to L_2 (but neither is parallel to the y-axis) then $m_2 m_1 = -1$, so the denominator of the right hand member of (1) vanishes, whence $\theta = 90^\circ$.

5. Illustrative Examples:

a. Problem 6, page 15

Solution: The sketch is given by



Now $m_{AB} = \frac{1 - 3}{3 - 1} = \frac{-2}{2} = -1$. Hence using the point slope form we obtain an equation for AB: $y - 3 = -1(x - 1)$ (using the point A) or $y - 3 = -x + 1$ and finally $x + y - 4 = 0$. Convince yourself that the same result would have been obtained by using the point B instead of A.

b. Problem 16, page 15

Solution: We are given the line L: $3x + 4y = 12$ and we wish to find m_L. We write L in the slope-intercept form:

$$4y = -3x + 12$$
$$y = \frac{-3}{4}x + 3$$

Comparing with $y = mx + b$ we see that $m_L = -\frac{3}{4}$.

c. Suppose we are given the points $A(a_1, a_2)$ and $B(b_1, b_2)$, where $a_1 \neq b_1$; let us determine all points $P(x, y)$ such that AP is perpendicular to BP. Now

$$m_1 = \text{slope of AP} = \frac{a_2 - y}{a_1 - x}$$

and

$$m_2 = \text{slope of BP} = \frac{b_2 - y}{b_1 - x}$$

Therefore AP is perpendicular to BP if and only if $m_1 m_2 = -1$, i.e., if and only if

$$\frac{(a_2 - y)(b_2 - y)}{(a_1 - x)(b_1 - x)} = -1$$

Simplifying this equation we obtain

$$x^2 + y^2 - (a_1 + b_1)x - (a_2 + b_2)y + (a_1 a_2 + b_1 b_2) = 0.$$

The graph of this equation is a circle (in fact, the line segment AB is a diameter of this circle). Therefore AP is perpendicular to BP if and only if P lies on the circle having AB as a diameter. A complete discussion of the circle is given in Chapter 4 of the text.

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Written Assignment

Solve Problems 1-25 (odd-numbered problems) and, in addition, Problem 22, a and b, pages 15 and 16 in the textbook.

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Hold this work until you are requested to mail it.

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Lesson 5

FUNCTION AND ABSOLUTE VALUE

Reference Material

Thomas, Chapter 1, Section 6

Supplementary Explanation

1. There are two important concepts in this section, the concept of function and the concept of absolute value.

2. As an alternate definition of function consider the following statement. A function is said to be defined when:

   a. A set D is given.

   b. To each element of D, one and only one object is assigned.

   The set D is called the domain of the function.

   Generally throughout the course the domain will be taken to be the set of real numbers. However many other domains are possible. Describe a function whose domain consists of elements which are not numbers. Convince yourself that the definition of function given here is equivalent to that given in the text.

   A function f may be specified by telling what the domain is and assigning, to each element of that domain, an object. The method of assigning may be an equation. For example if D is the set of real numbers X, then $f(x) = x^2 - 1$ will define a function. The function f associates with each real number x the real number $x^2 - 1$. For example, $f(0) = 0^2 - 1 = -1$, $f(1) = 1^2 - 1 = 0$; $f(3) = 3^2 - 1 = 8$; $f(-4) = (-4)^2 - 1 = 15$; $f(\sqrt{2}) = (\sqrt{2})^2 - 1 = 1$.

Do not confuse the symbols f and $f(x)$. The function is denoted by f while $f(x)$ denotes the value of the function at the number x.

A function cannot be defined by an equation such as $y = \pm\sqrt{x}$.

3. There are different types of functions. A constant function is described by $f(x) = c$ where x is any number and c is some fixed real number (Example $f(x) = 5$). A polynomial function is any function f defined by an equation of the form

$$f(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}\, x + a_n$$

where the $a_0, a_1, \dots, a_n$ are given real numbers and n is a given non-negative integer. The right member of the above equation is called a polynomial. We shall take as the domain of a polynomial function the set of all real numbers. (Examples of polynomial functions: f described by $f(x) = -2x + 5$, $f(x) = 4x^2 - 3x + 1$, $f(x) = 3x^5 + x$). A rational function is a function defined as a quotient of two polynomials. (Example: f described by $f(x) = \dfrac{2x^2 - 4x + 3}{x + 1}$ with domain the set of all real numbers $x \neq -1$.)

An algebraic function is one which is defined in terms of polynomials and roots of polynomials. (Example: $f(x) = \dfrac{x^3 + 1}{x\sqrt[4]{3x^2 + 2}}$ with domain the set of all non-zero real numbers.)

The six trigonometric functions, the logarithmic, and the exponential function are examples of non-algebraic functions and they are called transcendental functions.

4. The definition of the absolute value function is given at the bottom of page 19. Learn this definition. Verify, using this definition, that the following results are true.

$$|2| = 2,\ \left|\frac{1}{3}\right| = \frac{1}{3},\ |0| = 0,\ |-2| = 2,\ |\pi - 3| = \pi - 3,$$

$$\left|\pi - \frac{29}{7}\right| = \frac{29}{7} - \pi.$$

5. For the sake of completeness we shall list several properties of absolute value each of which may be proved rigorously.

   A.1 For each number a, $-|a| \leq a \leq |a|$

   A.2 If $d > 0$, then $|c| < d$ if and only if $-d < c < d$

   A.3 If $d \geq 0$, then $|c| \leq d$ if and only if $-d \leq c \leq d$

   A.3 $|a + b| \leq |a| + |b|$

   A.4 $|a^2| = |a|^2 = a^2$

   A.5 $|ab| = |a|\,|b|$

   A.6 $|-a| = |a|$

   A.7 $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, $b \neq 0$

   In the future we shall refer to these properties by their respective labels, A.1, A.2, etc. How many of these can you prove? (If you have trouble see your algebra text.) For a list of properties of inequalities and some properties of absolute values see Problems 47 and 48 on page 51 of Thomas

6. Here is an example which may help you. Solve the inequality $|3 - 2x| < 1$.

   Solution: By A.2 $|3 - 2x| < 1$ if and only if $-1 < 3 - 2x < 1$ or $-4 < -2x < -2$ or $2 > x > 1$. (Why?) The solution is the set of all real numbers x between 1 and 2.

   Problem: Solve the inequality $|x + 2| \geq 5$. Check your answer.

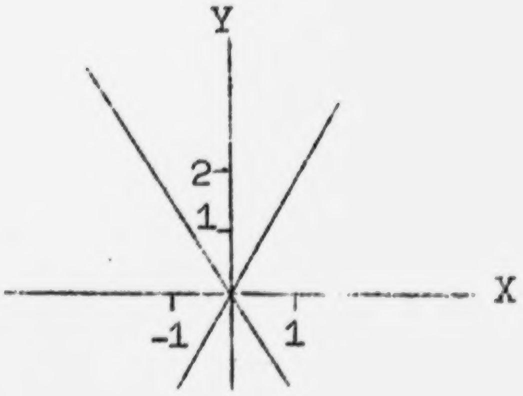
7. Definition: The graph of a function f is the graph of the equation $y = f(x)$. What is the graph of the function f defined by $f(x) = 2x + 3$?

8. The following are examples which illustrate certain techniques which may be used to graph various types of equations.

   a. Graph the equation

   $$(y - 2x)(y + 2x) = 0.$$

   Since a product is 0 if either factor is 0, it follows that P(x, y) will be a point on the graph of this equation if the coordinates of P satisfy either $y - 2x = 0$ or $y + 2x = 0$. Thus the graph of this equation consists of two straight lines, as shown in the figure.



Next, suppose we wish to graph the equation

$$(y - 2x)^2 + (y + 2x)^2 = 0.$$

Now $A^2 + B^2 = 0$ if and only if both A and B are 0, (for if $A \neq 0$, say, then $A^2 > 0$ so $B^2 = -A^2$ would be less than 0, which is impossible for any real number B). Therefore P(x, y) will be a point on the graph of this equation if and only if the coordinates of P satisfy both $y - 2x = 0$ and $y + 2x = 0$. Thus the graph of this equation consists of the (unique) point of intersection of these two lines, namely the origin.

b. Let us graph the equation

$$y = \frac{|x|}{x}.$$

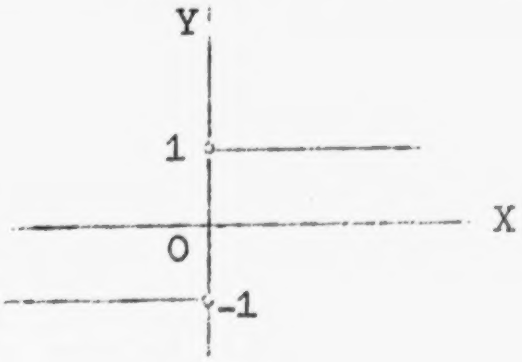
For $x > 0$ this equation becomes

$$y = \frac{x}{x} \text{ or } y = 1,$$

while for $x < 0$ it becomes

$$y = \frac{-x}{x} = -1.$$

When $x = 0$, y is clearly undefined. Therefore the graph is as shown below where the circles indicate that the points (0, 1) and (0, -1) do not lie on the graph.



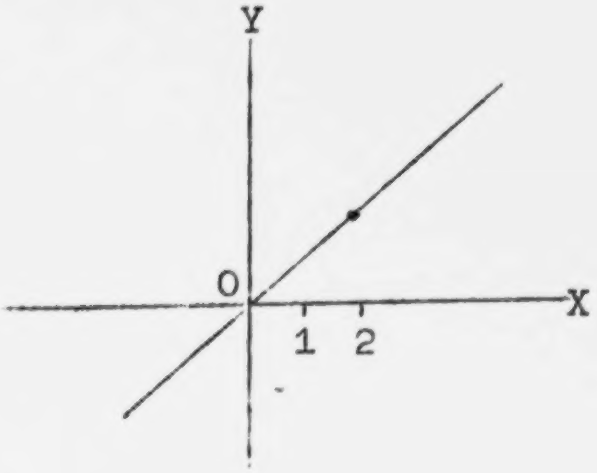
c. Let us graph the equation

$$y = \frac{x^2 - 2x}{x - 2}.$$

When $x \neq 2$, $x - 2 \neq 0$ so

$$y = \frac{x^2 - 2x}{x - 2} = \frac{x(x - 2)}{x - 2} = x.$$

However when $x = 2$, y is undefined. Therefore the graph of this equation is a straight line with a single point deleted as shown in the figure.



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Written Assignment

Solve Problems 1, 3, 5, 7, 8, 9, 10, 11, and 12, page 24 in the text.

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Give the prepared work for Lesson 4 and 5 to the supervisor who will complete the mailing procedure.

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Lesson 6

SLOPE AND DERIVATIVE

Reference Material

Thomas, Chapter 1, Sections 7 and 8

Supplementary Explanation

1. Section 7 deals with slope. Read and study this section very carefully. Be sure to know how one defines the slope of a curve at a point.

The following example will be given to supplement the one given in the text.

Problem 6, page 27

We are given $y = x^2 + 4x + 4$. Now if $P_1(x_1, y_1)$ is a point on this curve, then $y_1 = x_1{}^2 + 4x_1 + 4$ (Why?) and if $Q(x_2, y_2)$ is a second point on the curve, and if $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$ then $x_2 = x_1 + \Delta x$, $y_2 = y_1 + \Delta y$ and since also $y_2 = x_2{}^2 + 4x_2{}^2 + 4$, we have $y_1 + \Delta y = (x_1 + \Delta x)^2 + 4(x_1 + \Delta x) + 4$ or

$$\begin{aligned} y_1 + \Delta y &= x_1{}^2 + 2x_1\Delta x + (\Delta x)^2 + 4x_1 + 4\Delta x + 4 \text{ while} \\ y_1 &= x_1{}^2 \qquad\qquad\qquad\qquad\quad + 4x_1 \qquad\; + 4. \end{aligned}$$

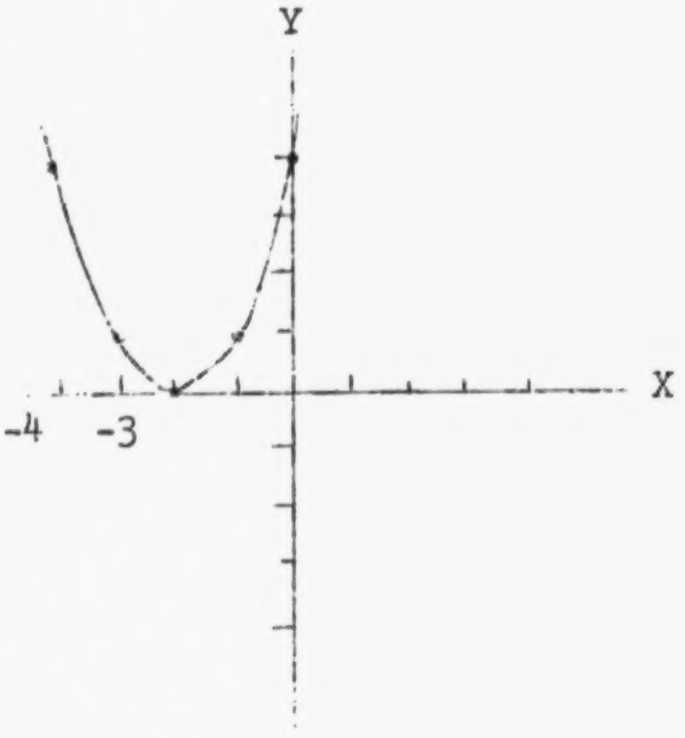
Subtracting the second equation from the first yields

$$\Delta y = 2x_1\Delta x + (\Delta x)^2 + 4\Delta x \text{ and thus}$$

$m_{sec} = \dfrac{\Delta y}{\Delta x} = 2x_1 + 4 + \Delta x$. Hence the slope of the curve at the point $P(x_1, y_1)$ = limit of m_{sec} as Δx approaches zero $= 2x_1 + 4$. Consequently the slope of the curve at any point is $m = 2x + 4$. Note immediately that the tangent to the curve is horizontal when $m = 0$ or when $x = -2$. Further $m < 0$ when $2x + 4 < 0$ or when $x < -2$, therefore $m > 0$ when $x > 2$. We may construct the following table.

x	y	m
-4		-4
-3	1	-2
-2	0	0
-1	1	2
0	4	4
1	9	6

The graph is shown in the figure on the following page.



2. Section 8 deals with the derivative of a function. First read and study this section very carefully. Note that the derivative of a function f represents the slope of the tangent line to the curve at any point (and hence the slope of the curve at any point). Further the derivative of f is defined as follows:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$ provided this limit exists.

What is the domain of f'?

The following notations for derivative are often used.

$$y' \qquad \frac{dy}{dx} \qquad f'(x) \qquad D_x y$$

Examples:

Problem 4, page 32

We are given $f(x) = x^2 - x + 1$ and we wish to find $f'(x)$. By definition

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$ Now for our problem

$$f'(x) = \lim_{\Delta x \to 0} \left[\frac{\{(x + \Delta x)^2 - (x + \Delta x) + 1\} - (x^2 - x + 1)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{x^2 + 2x\Delta x + (\Delta x)^2 - x - \Delta x + 1 - x^2 + x - 1}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{2x\Delta x + (\Delta x)^2 - \Delta x}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[2x - 1 + \Delta x \right]$$

$$= 2x - 1.$$

Notice that the procedure here is rather simpler than that used in solving the problem considered in part 1 above.

Problem 16, page 32

Since $f(x) = \sqrt{x + 1}$, we have

$$f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{(x + \Delta x) + 1} - \sqrt{x + 1}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{(x + \Delta x) + 1} - \sqrt{x + 1}}{\Delta x} \cdot \frac{\sqrt{(x + \Delta x) + 1} + \sqrt{x + 1}}{\sqrt{(x + \Delta x) + 1} + \sqrt{x + 1}}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x) + 1 - (x + 1)}{\Delta x(\sqrt{(x + \Delta x) + 1} + \sqrt{x + 1})}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x 1}{\Delta x(\sqrt{(x + \Delta x) + 1} + \sqrt{x + 1})}$$

$$= \frac{1}{2\sqrt{x + 1}}.$$

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Written Assignment

Solve Problems 1, 7, and 15, page 27 in the textbook. Solve Problems 1-19 (odd-numbered problems) on pages 32 and 33 in the textbook.

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Hold this work until you are requested to mail it.

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Lesson 7

VELOCITY, RATES AND LIMITS

Reference Material

Thomas, Chapter 1, Sections 9 and 10

Supplementary Explanation

1. We note from Section 9 that there are many other applications of the derivative concept such as instantaneous velocity at time t. Since the explanation is quite clear we shall go immediately into examples.

   Problem 10, page 37

   Given: $s = 64t - 16t^2$. Now from Problem 1, page 37 of the text we have (since $a = -16$, $b = 64$) $V = \frac{ds}{dt} = s'(t) = -32t + 64$.

   Problem 14, page 38

   Given: $V = \frac{4}{3}\pi r^3$. We want $\frac{dv}{dr}$. Now by definition

$$\frac{dv}{dr} = \lim_{\Delta r \to 0} \frac{\frac{4}{3}\pi(r + \Delta r)^3 - \frac{4}{3}\pi r^3}{\Delta r}$$

$$= \frac{4}{3}\pi \lim_{\Delta r \to 0} \frac{r^3 + 3r^2\Delta r + 3r(\Delta r)^2 + (\Delta r)^3 - r^3}{\Delta r}$$

$$= \frac{4}{3}\pi \lim_{\Delta r \to 0} \frac{\Delta r[3r^2 + 3r\Delta r + (\Delta r)^2]}{\Delta r}$$

$$= \frac{4}{3}\pi \cdot 3r^2 = 4\pi r^2.$$

2. Section 10 must be studied very carefully before one proceeds in the course since it involves perhaps the most basic concept in the calculus--the concept of limit. We have already been exposed to a special limit, the derivative, and have seen many of its applications. In this section we treat the concept of limit very rigorously.

   Note first, that the definition of limit (see page 40), may be rewritten as follows:

   > $\lim_{x \to a} f(x) = b$ if for every $\varepsilon > 0$ (real) there exists a $\delta > 0$ (real) such that $|f(x) - b| < \varepsilon$ whenever $0 < |x - a| < \delta$.

   Learn this definition (or the one in the text).

   To supplement the illustrations in the text we will consider the problems on the following pages.

a. Evaluate $\lim\limits_{x \to 1} \dfrac{x^2 + x - 2}{x - 1}$.

Solution: $\lim\limits_{x \to 1} \dfrac{x^2 + x - 2}{x - 1} = \lim\limits_{x \to 1} \dfrac{\cancel{(x - 1)}(x + 2)}{\cancel{(x - 1)}}$

$= \lim\limits_{x \to 1} (x + 2) = 3$.

Note that we could cancel x - 1 since x only approaches 1, it is never equal to 1.

b. Evaluate $\lim\limits_{x \to 3} (x^2 + x)$.

Solution: $\lim\limits_{x \to 3} (x^2 + x) \underset{\big\downarrow}{=} \lim\limits_{x \to 3} x^2 + \lim\limits_{x \to 3} x = 9 + 3 = 12$.

by Theorem 1 - (i), page 44

c. Prove $\lim\limits_{x \to 1} \dfrac{x^2 + x - 2}{x - 1} = 3$.

Proof: We must use the definition of limit i.e. We must prove that given any $\varepsilon > 0$. We can find a $\delta > 0$ such that

$$\left| \frac{x^2 + x - 2}{x - 1} - 3 \right| < \varepsilon \text{ whenever } \left| x - 1 \right| < \delta.$$

Consider

$$\left| \frac{x^2 + x - 2}{x - 1} - 3 \right| = \left| \frac{x^2 + x - 2 - 3(x - 1)}{x - 1} \right|$$

$$= \left| \frac{x^2 + x - 2 - 3x + 3}{x - 1} \right| = \left| \frac{x^2 - 2x + 1}{x - 1} \right| = \left| \frac{\cancel{(x - 1)}(x - 1)}{\cancel{x - 1}} \right|$$

$$= \left| x - 1 \right| \text{ if } x \neq 1.$$

Thus $\left| \dfrac{x^2 + x - 2}{x - 1} - 3 \right|$ will be less than ε if $\left| x - 1 \right| < \varepsilon$ x = 1 excluded. In this case $\delta = \varepsilon$.

d. Prove $\lim\limits_{x \to 3} x^2 + x = 12$.

Proof: We must prove that given any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left| x^2 + x - 12 \right| < \varepsilon \text{ whenever } \left| x - 3 \right| < \delta.$$

Now $|x^2 + x - 12| = |(x + 4)(x - 3)| \underset{\downarrow}{=} |x + 4|\,|x - 3|$

by property A.5 (Lesson 5)

Suppose we assume $|x - 3| < 1$ (hence 1 is a candidate for δ) then by property A.2 of Lesson 5 we have $-1 < x - 3 < 1$ or $2 < x < 4$ or $6 < x + 4 < 8$ or $-8 < x + 4 < 8$ and hence by A.2 again $|x + 4| < 8$. Thus we would have $|x^2 + x - 12| = |x + 4|\,|x - 3| < 8|x - 3|$. So $|x^2 + x - 12|$ will be less than ε if $8|x - 3| < \varepsilon$ or when $|x - 3| < \frac{\varepsilon}{8}$.

Now we notice that we have two candidates for δ, namely 1 and $\frac{\varepsilon}{8}$. If we choose δ to be the smaller of 1 and $\frac{\varepsilon}{8}$ then we can be sure that $|x^2 + x - 12| < \varepsilon$ whenever $|x - 3| < \delta$. Notice that in this example and in Example 1 on page 41 in the text δ is chosen to be the smaller of two real numbers. Precisely whys is such a choice permitted? (Please write for the answer to this question if you have any difficulty with its answer.)

3. Some further comments and results may be helpful.

   a. Note the difference between the problems of evaluation and proof (see Section 2 above).

   b. Study very carefully the results of Theorems 1 and 2 (pages 44 and 46). Make up some examples to illustrate the theorems.

   c. Understand clearly what is meant by $\lim\limits_{t \to c} F(t) = \infty$.

   d. You will be expected to prove that certain limits exist (study the book and the examples given in the previous section).

   e. Note that in the statement of Theorem 1 page 44 in the text, it is assumed that $\underset{t \to c}{\text{limit}}\, F_1(t)$ and $\underset{t \to c}{\text{limit}}\, F_2(t)$ both exist.

   To show that this restriction is necessary, consider the following situation.

   $\underset{x \to 0}{\text{limit}}\, \frac{1}{x}$ does not exist, but

   $\underset{x \to 0}{\text{limit}} \left(\frac{1}{x} - \frac{1}{x}\right) = \underset{x \to 0}{\text{limit}}\ 0 = 0$. Therefore,

   $\underset{x \to 0}{\text{limit}} \left(\frac{1}{x} - \frac{1}{x}\right) \neq \underset{x \to 0}{\text{limit}}\, \frac{1}{x} - \underset{x \to 0}{\text{limit}}\, \frac{1}{x}$.

f. Until now we have not really justified the use of the equality sign in the expression

$$\underset{x \to a}{\text{limit}}\ f(x) = b,$$

since as far as we know, the limit of f as x approaches a could have more than one value. To clarify this situation we shall prove that if

$$\underset{x \to a}{\text{limit}}\ f(x) = b \text{ and } \underset{x \to a}{\text{limit}}\ f(x) = c,$$

then $b = c$.

Proof: Suppose that $b \neq c$. Let $\varepsilon = \frac{1}{3}|b - c|$.

Then, since $\underset{x \to a}{\text{limit}}\ f(x) = b$, there is a number $\delta_1 > 0$ such that $|f(x) - b| < \varepsilon$ whenever $0 < |x - a| < \delta_1$.

Likewise, since $\underset{x \to a}{\text{limit}}\ f(x) = c$, there is a number $\delta_2 > 0$ such that $|f(x) - c| < \varepsilon$ whenever $0 < |x - a| < \delta_2$.

Now let δ denote the smaller of the numbers δ_1 and δ_2. Thus if $0 < |x - a| < \delta$ then $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$, so $|f(x) - b| < \varepsilon$ and $|f(x) - c| < \varepsilon$. That is, $|f(x) - b| < \varepsilon$, $|f(x) - c| < \varepsilon$ whenever $0 < |x - a| < \delta$.

Thus whenever $0 < |x - a| < \delta$ then

$$\begin{aligned} |b - c| &= |[f(x) - c] - [f(x) - b]| \\ &\leq |f(x) - c| + |f(x) - b| \quad \text{(by A.3 of Lesson 5)} \\ &< \varepsilon + \varepsilon \\ &= 2\varepsilon \\ &= \tfrac{2}{3}|b - c|. \end{aligned}$$

Therefore $|b - c| < \frac{2}{3}|b - c|$ or, subtracting $\frac{2}{3}|b - c|$

from both members, $\frac{1}{3}|b - c| < 0$. But by definition of absolute value this is impossible. Thus our assumption that $b \neq c$ must be false, so $b = c$.

* * * * *

Written Assignment

Solve Problems 1-15 (odd-numbered problems) on pages 37 and 38 in the textbook. Solve Problems 1-9 on pages 46 and 47 in the textbook.

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Give the prepared work for Lessons 6 and 7 to the supervisor who will complete the mailing procedure.

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Note: In preparation for your first examination you should solve as many of the miscellaneous problems found on pages 48-51 as you feel necessary.

We present the solution to Problem 8 on page 48. Instead of solving this problem directly we shall derive the so-called normal equation of a line L and as a consequence obtain the result desired in the problem.

Let P be the projection of the origin O onto L (if L passes through O, then $p = 0$), and let $|OP| = p$. Let ω denote the smallest non-negative angle from the positive x-axis to the line segment OP (where, if $O = P$, we take ω to be $\alpha + 90^\circ$, where α is the inclination of L). Some possibilities for L, P, p, and ω are shown in Figure 1.

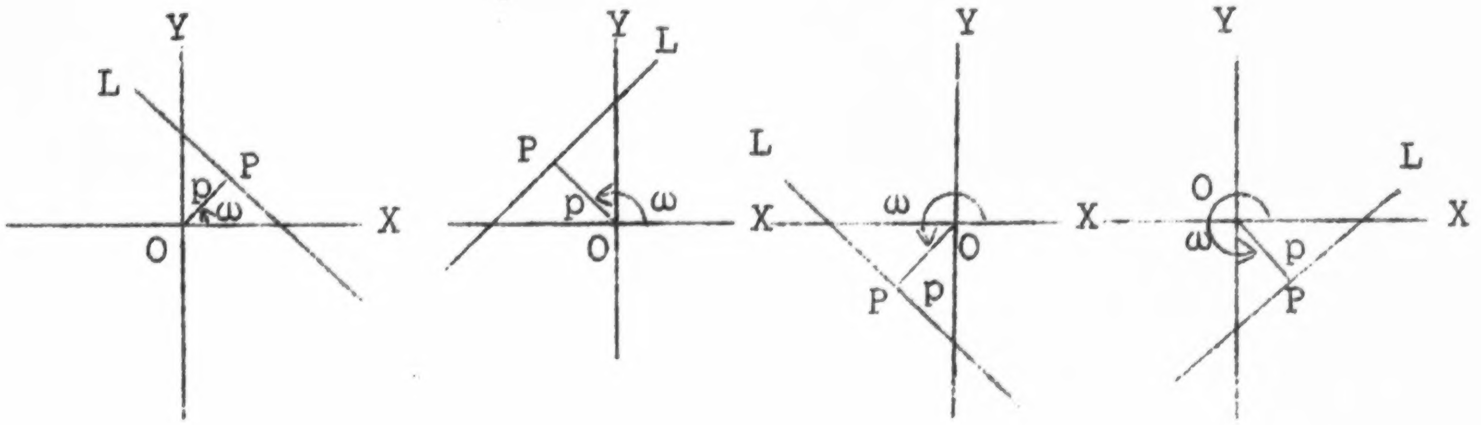

Figure 1

Then if P has coordinates (x, y), we see that $\cos \omega = x/p$ and $\sin \omega = y/p$, so

$$P = (x, y) = (p \cos \omega, p \sin \omega).$$

Moreover the line OP has slope equal to $\tan \omega$ so, since L is perpendicular to OP, L has slope

$$m = \frac{-1}{\tan \omega} = -\frac{\cos \omega}{\sin \omega}.$$

Therefore the point-slope form of the equation of L is

$$y - p \sin \omega = -\frac{\cos \omega}{\sin \omega}(x - p \cos \omega).$$

This reduces to

$$y \sin \omega - p \sin^2 \omega + x \cos \omega - p \cos^2 \omega = 0$$

or

$$x \cos \omega + y \sin \omega = p \sin^2 \omega + p \cos^2 \omega = p.$$

Thus L has equation

$$x \cos \omega + y \sin \omega - p = 0. \tag{1}$$

Equation (1) is called the normal equation of the line L. Note that $p \geq 0$ for every line L, since $p = |OP|$.

Example: Write the normal equations of the two lines each of which is 5 units from the origin and has inclination 45°. [See Figure 2.]

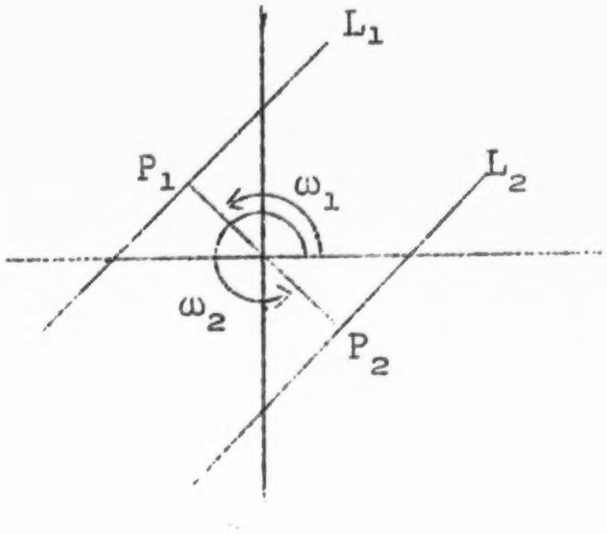

Figure 2

Denoting the lines by L_1 and L_2, and the corresponding constants in their normal equations by p_1, ω_1 and p_2, ω_2, we have that $p_1 = p_2 = 5$, $\omega_1 = 135^\circ$, $\omega_2 = 315^\circ$. Therefore the normal equation of L_1 is

$$x \cos 135^\circ + y \sin 135^\circ - 5 = 0,$$

i.e.,

$$-\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} - 5 = 0,$$

while the normal equation of L_2 is

$$x \cos 315^\circ + y \sin 315^\circ - 5 = 0,$$

i.e.,

$$\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} - 5 = 0.$$

We now show how to find the normal equation of a line L when another form of an equation for L, say

$$ax + by + c = 0,$$

is given. Then if K is any non-zero real number, L also has equation

$$(2) \qquad kax + kby + kc = 0.$$

We wish to determine k so that equations (1) and (2) will be identical. For this to be true, we must have

$$ka = \cos \omega, \; kb = \sin \omega, \; kc = -p.$$

Squaring and adding the first two of these equations, we obtain

$$k^2(a^2 + b^2) = \cos^2 \omega + \sin^2 \omega = 1;$$

therefore

$$k = \frac{1}{\pm\sqrt{a^2 + b^2}},$$

so

$$(3) \quad \cos \omega = \frac{a}{\pm\sqrt{a^2 + b^2}}, \; \sin \omega = \frac{b}{\pm\sqrt{a^2 + b^2}}, \; -p = \frac{c}{\pm\sqrt{a^2 + b^2}}.$$

But $p \geq 0$ so

$$\frac{c}{\pm\sqrt{a^2 + b^2}} = -p \leq 0.$$

Thus the sign of the radical must be unlike the sign of C.

Summarizing: If the line L has equation

$$ax + by + c = 0,$$

then L has normal equation

$$\frac{a}{\pm\sqrt{a^2 + b^2}}x + \frac{b}{\pm\sqrt{a^2 + b^2}}y + \frac{c}{\pm\sqrt{a^2 + b^2}} = 0, \text{ or} \tag{4}$$

$$\frac{ax + by + c}{\pm\sqrt{a^2 + b^2}} = 0$$

with the sign of the radical taken to be unlike the sign of c.

Thus for example if L is given by the equation $2x - 3y + 6 = 0$, then L has normal equation

$$\frac{2x - 3y + 6}{-\sqrt{13}} = 0 \text{ or } \frac{-2x}{\sqrt{13}} + \frac{3y}{\sqrt{13}} - \frac{6}{\sqrt{13}} = 0.$$

Solution to Problem 8, page 48:

Suppose that L is any line, with normal equation

$$x \cos \omega + y \sin \omega - p = 0,$$

and that $P_1(x_1, y_1)$ is any point not on L. [See Figure 3]

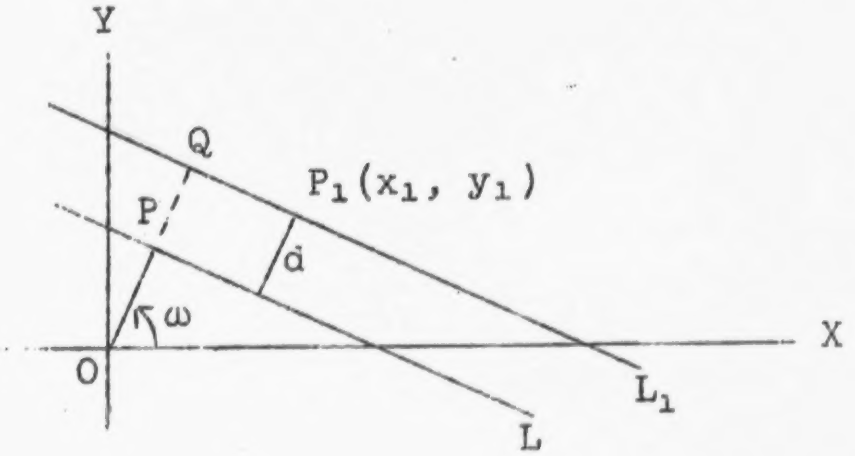

Figure 3

Then L_1 has normal equation

$$x \cos \omega + y \sin \omega - p_1 = 0,$$

where $p_1 = |OQ|$.. But $|OQ| = |OP| + |PQ| = p + d$, where d denotes the distance from P_1 to L. Therefore the normal equation L_1 is

$$x \cos \omega + y \sin \omega - (p + d) = 0.$$

Since $P_1(x_1, y_1)$ lies on L_1, its coordinates must satisfy this equation, so

$$x_1 \cos \omega + y_1 \sin \omega - (p + d) = 0.$$

Therefore

$$d = x_1 \cos \omega + y_1 \sin \omega - p. \tag{5}$$

Consequently the distance from a point to a line can be found by writing the normal equation of the line and substituting the coordinates of the point in the left member of this equation. [The sign of d, as given by (5), will be + or - according to whether P and O lie on opposite sides or the same side of L.]

Now using the results given by (3) (and assuming that we desire absolute distance) we have from (5):

$$d = \left|\frac{ax_1 + by_1 + c}{\pm\sqrt{a^2 + b^2}}\right| = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}. \tag{6}$$

Example: Find the distance d, from the point (3, 4) to the line $7x + y - 10 = 0$.

Solution: using (6) we have

$$d = \frac{7(3) + 4 - 10}{\sqrt{49 + 1}} = \frac{15}{\sqrt{50}} = \frac{3\sqrt{2}}{2}$$

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When the student has received the evaluated Written Assignment for Lessons 6 and 7 and has reviewed the work, he may request the supervisor to administer the hour examination.

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Lesson 8

POLYNOMIAL FUNCTIONS, RATIONAL FUNCTIONS AND THEIR DERIVATIVES

Reference Material

Thomas, Chapter 2, Sections 1 and 2

Supplementary Explanation

1. Study carefully the material in this Section 1 and note the following results: (we shall use one of the alternate notations for derivative, see page 30 of the textbook)

$D_xC = 0$, where C is any constant; $D_x x^n = nx^{n-1}$; where n is any positive integer; $D_x cu = cD_x u$; $D_x cx^n = cnx^{n-1}$;

$D_x[u_1 + u_2 + . . . u_n] = D_x u_1 + D_x u_2 + . . . + D_n u.$

Learn these results not only in terms of symbols but also in words.

2. Notation for higher derivatives will now be considered. The second derivative is given by $D_x y' = \frac{dy'}{dx} = \frac{d^2y}{dx^2} = D_x^2 y = f''(x) = y''$.

   Appropriate changes are made for higher derivatives. Recall that if $s = f(t)$ the instantaneous velocity any time t is defined as $v = \frac{ds}{dt}$. Similarly the instantaneous acceleration at any time t is defined as $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

   Examples:

   a. Problem 12, page 59

      We are given $y = 3x^7 - 7x^3 + 21x^2$ and are to find $D_x y$ and $D_x^2 y$.

      $$\begin{aligned} \text{Now}\quad D_x y &= D_x(3x^7 - 7x^3 + 21x^2) \\ &= D_x 3x^7 + D_x(-7x^3) + D_x 21x^2 \\ &= 3D_x x^7 - 7D_x x^3 + 21D_x x^2 \\ &= 3 \cdot 7x^6 - 7 \cdot 3x^2 + 21 \cdot 2x \\ &= 21x^6 - 21x^2 + 42x. \end{aligned}$$

      (Note the results, given in the first explanation, which were used in the solution of this problem.)

   b. Problem 20, page 59

      Given: the curve $y = ax^2 + bx + c$ and that it passes through (1, 2). Therefore, we know $2 = a(1)^2 + b(1) + c$ (Why?), in other words, $a + b + c = 2$. Now if this curve is to be tangent to the line $y = x$ at the orgin then their slopes must be equal at the origin (0, 0) i.e. $m_{curve} = m_{tangent\ line}$ at (0, 0).

      Now $m_{curve} = D_x y = 2ax + b$ so m_{curve} at origin $= 2 \cdot a \cdot 0 + b = b$. The slope of the tangent line $m_{tangent\ line} = D_x y = D_x x = 1$ and hence $\underline{b = 1}$.

We further know that the curve passes through (0, 0) (Why do we know this?), therefore $0 = a \cdot 0 + b \cdot 0 + c$ and hence $\underline{c = 0}$. Finally since $a + b + c = 2$ we have $\overline{a + 1 + 0 = 2}$ or $\underline{a = 1}$.

3. Study carefully this Section 2 and make a list of the key results. Close attention should be paid to the results concerning the derivative of a product, quotient and power of a function. Learn all of the formulas (in words as well as in symbols).

4. Supplementary Exercises:

Problem 2, page 65

Given: $y = (x - 1)^3 (x + 2)^4$ to find $D_x y$.

Solution:

Tools to be used: $D_x uv = uD_x v + vD_x u$ and $D_x u^n = nu^{n-1} D_x u$.

[Convince yourself that these results are equivalent to (1) and (4) on pages 60 and 63 respectively.] Thus we have

$$\begin{aligned}
D_x y &= D_x[(x - 1)^3 (x + 2)^4] \\
&= (x - 1)^3 D_x(x + 2)^4 + (x + 2)^4 D_x(x - 1)^3 \\
&= (x - 1)^3 \cdot 4(x + 2)^3 D_x(x + 2) \\
&\quad + (x + 2)^4 \cdot 3(x - 1)^2 D_x(x - 1) \\
&= 4(x - 1)^3 (x + 2)^3 (D_x x + D_x 2) \\
&\quad + 3(x + 2)^4 (x - 1)^2 (D_x x - D_x 1) \\
&= 4(x - 1)^3 (x + 2)^3 (1 + 0) \\
&\quad + 3(x + 2)^4 (x - 1)^2 (1 - 0) \\
&= 4(x - 1)^3 (x + 2)^3 + 3(x + 2)^4 (x - 1)^2 \\
&= (x + 2)^3 (x - 1)^2 [4(x - 1) + 3(x + 2)] \\
&= (x + 2)^3 (x - 1)^2 (7x + 2).
\end{aligned}$$

Problem 6, page 66

Given: $y = \dfrac{2x + 1}{x^2 - 1}$ to find $D_x y$.

Solution:

Tools to be used, at least: $D_x\left(\frac{u}{v}\right) = \frac{vD_xu - uD_xv}{v^2}$

and $D_xx^n = nx^{n-1}$ and others. Thus we have

$$D_xy = D_x\left[\frac{2x + 1}{x^2 - 1}\right] = \frac{(x^2 - 1)\ D_x\ (2x + 1) - (2x + 1)\ D_x\ (x^2 - 1)}{(x^2 - 1)^2}$$

$$= \frac{(x^2 - 1)\ (2) - (2x + 1)\ (2x)}{(x^2 - 1)^2}$$

$$= \frac{2x^2 - 2 - 4x^2 - 2x}{(x^2 - 1)^2} = \frac{-2x^2 - 2x - 2}{(x^2 - 1)^2}$$

$$= \frac{-2\ (x^2 + x + 1)}{(x^2 - 1)^2}$$

Problem 10, page 66

Given: $s = (2t + 3)^3$ to find $\frac{ds}{dt}$.

Solution:

Tool to be used: $D_xu^n = nu^{n-1}\ D_xu$. Thus

$$\frac{ds}{dt} = D_ts = D_t\ (2t + 3)^3 = 3(2t + 3)^2 \cdot D_t(2t + 3)$$

$$= 3(2t + 3)^2\ (2)$$

$$= 6(2t + 3)^2.$$

(Note that in this problem we let $u = 2t + 3$.)

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Written Assignment

Solve Problems 1-21 (odd-numbered problems), page 59 in the textbook. Solve Problems 1-15 (odd-numbered problems), pages 65 and 66 in the textbook.

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Hold this work until you are requested to mail it.

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Lesson 9

IMPLICIT RELATIONS, THEIR DERIVATIVES AND INCREMENTS

Reference Material

Thomas, Chapter 2, Sections 3 and 4

Supplementary Explanation

1. Before proceeding with the problems associated with Section 3 one should have a clear notion of the difference between an *explicit function*, *implicit relation* and *implicit function*. If the difference between these notions is not clear then read once again the beginning of this section.

   The key tool used in finding $\frac{dy}{dx}$ from an implicit relation is the following result (see (4) and (5) on pages 63 and 64).

   If $u = g(x)$ is a differentiable function of x and n is any integer, then

   $\frac{du^n}{dx} = nu^{n-1} \frac{du}{dx}$ or written in equivalent notation

   $$D_x u^n = nu^{n-1} D_x u. \tag{1}$$

   Examples of the use of (1):

   a. $D_x(1 - 2x)^{-5} = -5(1 - 2x)^{-6} D_x(1 - 2x) = -5(1 - 2x)^{-6} (-2)$
   $= 10(1 - 2x)^{-6}$ (here $u = 1 - 2x$, and $n = -5$).

   b. $D_x x^7 = 7x^6 D_x x = 7x^6$ (Since $D_x x = 1$; here $u = x^7$ and $n = 7$.)

   c. $D_x Z^{-2} = -2Z^{-3} D_x Z$ (here $u = Z$, and $n = -2$)

   d. $D_x y^3 = D_x y^3 = 3y^2 \cdot D_x y = 3y^2 D_x y = 3y^2 \frac{dy}{dx}$
   (here $u = y$ and $n = 3$)

   Study these examples very carefully, especially d.

2. We now illustrate the method used in finding derivatives of implicit relations.

   Example 1: Problem 4, page 7

   Given $x^2y + xy^2 = 6$, find $\frac{dy}{dx}$.

   Solution:

   If $x^2y + xy^2 = 6$ then differentiating both sides with respect to x we have $D_x[x^2y + xy^2] = D_x 6$, or

   $D_x(x^2y) + D_x(xy^2) = 0$ (Why?) and so (since x^2y and xy^2 are products) we have by the product rule (see page 60 of the textbook):

   $x^2 D_x y + y D_x x^2 + x D_x y^2 + y^2 D_x x = 0$, and by (1) above

$x^2 D_x y + y \cdot 2x D_x x + x \cdot 2y D_x y + y^2 \cdot 1 = 0$ and this becomes

$(x^2 + 2xy)\, D_x y + 2xy + y^2 = 0$ and finally

$$D_x y = \frac{-(2xy + y^2)}{x^2 + 2xy}.$$

Now since $D_x y = \frac{dy}{dx}$ we have solved the problem.

Example 2: If $y^2 = 2px$ (where p is a fixed real number), find $\frac{d^2y}{dx^2}$ when $y \neq 0$.

Differentiating both sides of the given equation with respect to x, we obtain

$$D_x y^2 = D_x 2bx,$$

or

$$2y \cdot \frac{dy}{dx} = 2p,$$

(2)
$$y \cdot \frac{dy}{dx} = p.$$

Differentiating both members of (2) with respect to x, we obtain

(3)
$$y \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{dy}{dx} = 0.$$

[Notice that $y \cdot dy/dx$ is a product; hence we again apply the result of page 60.] Solving (2) for dy/dx gives $dy/dx = p/y$; substituting this expression for dy/dx in (3) yields

$$y \frac{d^2y}{dx^2} + \frac{p}{y} \cdot \frac{p}{y} = 0$$

or

$$\frac{d^2y}{dx^2} = -\frac{p^2}{y^3}.$$

Example 3: If $x^2 + 4y^2 = 25$, find the values of dy/dx and d^2y/dx^2 when $x = 3$, $y = 2$.

Differentiating the given equation term by term with respect to x, we obtain

(4)
$$2x + 8y \frac{dy}{dx} = 0.$$

Differentiating (4) term by term with respect to x,

$$(5) \qquad 2 + 8y \cdot \frac{d^2y}{dx^2} + 8 \cdot \frac{dy^2}{dx} = 0.$$

Thus

$$(6) \qquad \frac{d^2y}{dx^2} = \frac{-8\left(\frac{dy}{dx}\right)^2 - 2}{8y} .$$

Now from (4) we have

$$(7) \qquad \frac{dy}{dx} = \frac{-2x}{8y} = \frac{-x}{4y} .$$

Hence when x = 3, y = 2 we have from (7)

$$(8) \qquad \frac{dy}{dx} = \frac{-3}{4.2} = \frac{-3}{8}$$

and from (8) and (6) we have

$$\frac{d^2y}{dx^2} = \frac{-8\left(\frac{-3}{8}\right)^2 - 2}{8(2)} = \frac{-25}{128}.$$

3. No additional comments will be made concerning the results of Section 4. We shall supplement the illustrative examples with the following problems.

Problem 2, page 75

We are given $y = 2x^2 + 4x - 3$ to find (a) Δy.

We argue as follows

$$\begin{aligned} y + \Delta y &= 2(x + \Delta x)^2 + 4(x + \Delta x) - 3 \\ &= 2(x^2 + 2x\Delta x + (\Delta x)^2) + 4x + 4\Delta x - 3 \\ &= 2x^2 + 4x\Delta x + 2(\Delta x)^2 + 4x + 4\Delta x - 3. \end{aligned}$$

Since $y = 2x^2 + 4x - 3$, we have by subtraction

(a) $\Delta y = 4x\Delta x + 4\Delta x + 2(\Delta x)^2 = (4x + 4)\Delta x + 2(\Delta x)^2$.

Now for part (b) we have

$$\Delta y_{tan} = \frac{dy}{dx}\Delta x = (4x + 4)\Delta x$$

Finally for part (c):

$$\Delta y - \Delta y_{tan} = (4x + 4)\Delta x + 2(\Delta x)^2 - (4x + 4)\Delta x = 2(\Delta x)^2.$$

Problem 10, page 75

Given $y = \frac{x}{x + 1}$, $x = 1$, $\Delta x = 0.3$, we want to estimate

$y + \Delta y = f(x + \Delta x)$ by calculating $y + \frac{dy}{dx}\Delta x$.

Now $\frac{dy}{dx} = +\frac{1}{(x + 1)^2}$ (Check this!) so

$y + \frac{dy}{dx}\Delta x = \frac{x}{x + 1} + \frac{1}{(x + 1)^2}\Delta x$. Now when $x = 1$, and

$\Delta x = 0.3$ this becomes

$$y + \frac{dy}{dx}\Delta x = \frac{1}{2} + \frac{1}{(2)^2}(0.3) = \frac{1}{2} + \frac{3}{40} = \frac{23}{40}.$$

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Written Assignment

Solve Problems 1-34, page 70, do multiples of 3 only. Solve Problems 1-11, page 75, do odd-numbered problems only.

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Give the prepared work for Lessons 8 and 9 to the supervisor who will complete the mailing procedure.

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Lesson 10

THE CHAIN RULE AND DIFFERENTIAL

Reference Material

Thomas, Chapter 2, Sections 5, 6, and 7

Supplementary Explanation

1. As you have observed, the notion of parametric equations and equations (3b), (5) and (6) are the key results of Section 5. Note well their origin and have them available for immediate use.

   We shall consider the following examples:

   Example 1: Problem 2, page 79

   a. Since $x = t^2$ and $y = t^3$, we have $y = x^{\frac{3}{2}}$.

   b. From $y = x^{\frac{3}{2}}$ follows $\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}t$ and

      since $x = t^2$ and $y = t^3$, we have $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2$.

   c. From (3b) on page 76 of the textbook we know $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$ and this result holds for our problem since

      $3t^2 = (\frac{3}{2}t)(2t)$.

   Example 2: Problem 8, page 80

   a. Since $y = Z^{\frac{2}{3}}$ and $Z = x^2 + 1$ and we know from equation (6) on page 79 (with u replaced by Z):

      $\frac{dy}{dx} = \frac{dy}{dZ} \cdot \frac{dZ}{dx}$ We may argue as follows

$$\frac{dy}{dZ} = \frac{2}{3}Z^{-\frac{1}{3}},\ \frac{dZ}{dx} = 2X,\ \text{therefore}$$

$$\frac{dy}{dx} = \frac{2}{3}Z^{-\frac{1}{3}} \cdot 2x = \frac{4x}{3Z^{-\frac{1}{3}}} = \frac{4x}{3(x^2 + 1)^{\frac{1}{3}}}$$

b. Now $y = Z^{\frac{2}{3}}$, $Z = x^2 + 1$ imply that $y = (x^2 + 1)^{\frac{2}{3}}$. Therefore, using the result stated in page 68 of the textbook we have:

$$D_x y = D_x(x^2 + 1)^{\frac{2}{3}} = \frac{2}{3}(x^2 + 1)^{-\frac{1}{3}} \cdot D_x(x^2 + 1)$$

$$= \frac{2}{3}(x^2 + 1)^{-\frac{1}{3}}\ (2x) = \frac{4x}{3(x^2 + 1)^{\frac{1}{3}}}$$

2. One should observe that the curve represented parametrically may not necessarily be identical to the curve obtained by eliminating the parameter. For example if $x = \sqrt{2 + t}$, $y = \sqrt{2 - t}$ then observe that x and y are both positive. Thus the graph would lie in the first quadrant. However, upon eliminating the parameter one obtains the equation $x^2 + y^2 = 4$ which is a circle with center at (0, 0) and radius 2. It will lie in every quadrant.

3. Now let us consider the middle of page 80. Differentials are useful in approximating functions and in finding derivatives, especially if a curve is described by parametric equations. As a result of this section we may think of $\frac{dy}{dx}$ as the ratio of two differentials and $\frac{d^2y}{dx^2} = \frac{dy'}{dx}$ again as the quotient of two differentials.

   Several examples will be given to illustrate these ideas. After studying these examples read Sections 6 and 7 once again before attempting the homework. Note carefully the formulas on page 85.

   Example 1: Problem 2, page 86

   Given: $y^2 = (3x^2 + 1)^{\frac{3}{2}}$. Taking the differential of both sides we have

$$dy^2 = d(3x^2 + 1)^{\frac{3}{2}} \text{ or } 2ydy = \frac{3}{2}(3x^2 + 1)^{\frac{1}{2}}\ 6x\ dx \text{ or}$$

$$dy = \frac{9x}{2y}(3x^2 + 1)^{\frac{1}{2}}dx$$

Example 2: Problem 12, page 86

Let $y = f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$, so $dy = \frac{1}{3x^{\frac{2}{3}}}\,dx$.

We want $\sqrt[3]{0.126}$. If we let $x = .125$, and $dx = 0.001$, then our approximation to $\sqrt[3]{0.126}$ is obtained by evaluating

$$y + dy = \sqrt[3]{0.125} + \frac{1}{3(0.125)^{\frac{2}{3}}}\,0.001$$

$$= 0.5 + \frac{0.001}{3(.5)^2} = 0.5 + \frac{.1}{3000(.25)}$$

$$= 0.5 + \frac{1}{750} = 0.5013.$$

Example 3: Problem 26, page 87

Given: $x = \frac{t-1}{t+1}$, $y = \frac{t+1}{t-1}$. Thus:

$$dx = \frac{[(t+1)(1) - (t-1)(1)]}{(t+1)^2}\,dt = \frac{[t+1-t+1]}{(t+1)^2}\,dt$$

$$dx = \frac{2}{(t+1)^2}\,dt = 2(t+1)^{-2}dt$$

$$dy = \frac{[(t-1)(1) - (t+1)(1)]}{(t-1)^2}\,dt = \frac{[t-1-t-1]}{(t-1)^2}\,dt$$

$$dy = -2(t-1)^{-2}\,dt.$$

Hence: $\frac{dy}{dx} = \frac{-2(t-1)^{-2}\,dt}{2(t+1)^{-2}\,dt} = -\left[\frac{(t+1)}{(t-1)}\right]^2$

Now $\frac{d^2y}{dx^2} = \frac{dy'}{dx}$. We know $dx = 2(t+1)^{-2}\,dt$, so we must find dy'. From above we have

$y' = -\left[\frac{t+1}{t-1}\right]^2$. It then follows that

$$dy' = -2\left[\frac{t+1}{t-1}\right]\left[\frac{(t-1)(1) - (t+1)\ 1}{(t-1)^2}\right]dt$$

$$= +4\frac{(t+1)}{(t-1)^3}\,dt$$

Consequently,

$$\frac{d^2y}{dx^2} = \frac{4\,\frac{t+1}{(t-1)^3}\,dt}{2\,(t+1)^{-2}\,dt} = \frac{2(t+1)^3}{(t-1)^3}.$$

Example 4: Problem 21, page 87

[This problem is assigned. Solve it now and then check your solution with the one that is given here.]

We are given the parametric equations $x = f(t)$, $y = g(t)$.

By (5), page 78, we have therefore $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$, $x'(t) \neq 0$.

Hence, making use of (6) on page 79, with $u = t$ and the quotient rule for derivatives,

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dt} \cdot \frac{dt}{dx} = \frac{d\left(\frac{y'(t)}{x'(t)}\right)}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{x'(t) \cdot y''(t) - y'(t) \cdot x''(t)}{(x'(t))^2} \cdot \frac{dt}{dx}.$$

Since $\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}} = \frac{1}{x'(t)}$, we obtain therefore

$$\frac{d^2y}{dx^2} = \frac{x'(t) \cdot y''(t) - y'(t) \cdot x''(t)}{(x'(t))^3}, \quad x'(t) \neq 0,$$

which is the same as the result desired.

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Written Assignment

Solve Problems 1-9, odd-numbered problems, pages 79 and 80.
Solve Problems 1-21, odd-numbered problems, pages 86 and 87.

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Hold this work until you are requested to mail it.

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Lesson 11

CONTINUITY

Reference Material

Thomas, Chapter 2, Section 8

Supplementary Explanation

1. As was asserted previously the limit concept is one of the most important concepts in the calculus. The derivative of a function at a point represented the first extremely useful example of a limit. Now we have the second example

which yields a rigorous definition of the concept of continuity of a function at a point (or in or on an interval).

2. The definition of continuity of a function f at a point x = c (in the domain of f), as given on page 87, is equivalent to checking whether or not all three of the following statements are true.

   a. f(c) exists

   b. $\lim_{x \to c} f(x)$ exists

   c. $\lim_{x \to c} f(x) = f(c)$

   Suggestion: given an ε, δ definition of continuity of a function f at the point x = c.

   Example 1: Prove that the function described by $f(x) = x^3$ is a continuous at x = 4.

   Solution: We must check to see whether or not the conditions a, b, and c given above are satisfied.

   a. $f(4) = 4^3 = 64$, $\therefore$ f(4) exists

   b. $\lim_{x \to 4} x^3 = 64$, $\therefore \lim_{x \to 4} f(x) = \text{limit}_{x \to 4}\, x^3$ exists

   c. Finally we note $\lim_{x \to 4} f(x) = f(4)$

   Conclusion: f is a continuous at x = 4. Is this function continuous at all points of the interval $-2 \leq x \leq 6$?

3. A function f is said to be discontinuous at the number c if f is not continuous at c. Because of the definition of continuity, f may be discontinuous at c if any one of a, b, or c is not satisfied.

   Example 2: Let f be defined by $f(x) = \frac{x^2 - 2x}{x - 2}$.

   Is f continuous at 2?

   Solution: Checking, we find

   a. f(2) does not exist. (Why?) Hence f is discontinuous at 2. (There is no need to also check b and c.)

Example 3: Let f be defined by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Is f continuous at 0?

Solution: Checking, we find

a. $f(0) = 0$ (exists)

b. $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \frac{1}{x} = \infty$ does not exist!

Hence f is discontinuous at 0.

Example 4: Let f be defined by

$$f(x) = \begin{cases} \frac{x^2 - 2x}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2. \end{cases}$$

Is f continuous at 2?

Solution: Checking, we find

a. $f(2) = 1$, exists!

b. $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2} \frac{x^2 - 2x}{x - 2} = \lim\limits_{x\to 2} \frac{x(x - 2)}{x - 2}$

$= \lim\limits_{x\to 2} x = 2$, exists!

(Here we may divide numerator and denominator by $x - 2$ without fear of dividing by zero since by definition of limit, the value of the $\lim\limits_{x\to 2} f(x)$ does not depend upon the value of $f(x)$ when x is equal to 2, but only on values of $f(x)$ when x is close to but different from 2.)

c. However, we note that

$\lim\limits_{x\to 2} f(x) \neq f(2)$.

Consequently, f is discontinuous at 2.

How would you redefine the function in this example so that it would be continuous at $x = 2$? Sketch a graph of f.

4. This section concludes with three important theorems each of which will be referred to often in the future. We have found that it is rather easy to find the derivative of many

different functions; as a consequence of Theorem 1 on page 90 we have a simple way of determining continuity of these functions (without using the definition). Remember that the converse of Theorem 1 is not true. The classical example is $f(x) = |x|$ at $x = 0$, see page 91. This function is continuous at $x = 0$ but not differentiable at $x = 0$. By the way, the argument given at the bottom of page 91 that the $\lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x}$ does not exist is not rigorous. To prove this statement rigorously requires the use of the ε, δ definition of limit. The proof of this result may be given as follows:

Assume $\lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x} = L$ i.e. it exists. Thus from the definition of limit we know that for every $\varepsilon > 0$ we can find a $\delta > 0$ such that

$\left| \frac{|\Delta x|}{\Delta x} - L \right| < \varepsilon$ whenever $0 < |\Delta x - 0| < \delta$ or (again see A.2, page 20 of these notes) $L - \varepsilon < \frac{|\Delta x|}{\Delta x} < L + \varepsilon$ whenever $-\delta < \Delta x < \delta$.

Suppose $\varepsilon = 1$, then there must be some $\delta > 0$ such that

(1) $\left(L - 1 < \frac{|\Delta x|}{\Delta x} < L + 1 \text{ whenever } -\delta < \Delta x < \delta\right)$.

Now since $\Delta x \neq 0$, we note that if $\Delta x > 0$, $\frac{|\Delta x|}{\Delta x} = 1$ and from (1) it follows that

(2) $$1 < L + 1.$$

If $\Delta x < 0$, then $\frac{|\Delta x|}{\Delta x} = -1$ (Why?) and from (1) it follows that

(3) $$L - 1 < -1.$$

Thus we have from (2) that $L > 0$ and from (3) that $L < 0$. Since this is impossible we conclude that our assumption that $\lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x}$ exists is false.

5. Study the statements of Theorems 2 and 3, page 92, very carefully so that they may be available when needed.

6. Consider the following example: Prove that the function $F(x) = x^3(x^2 - 1)^{\frac{1}{2}}$ is continuous at $x = 2$.

   Solution: Let $f(x) = x^3$ and $g(x) = (x^2 - 1)^{\frac{1}{2}}$. Now since both $f'(2)$ and $g'(2)$ exist (check this) we know by Theorem 1 that $f(x)$ and $g(x)$ are both continuous at $x = 2$. Consequently from Theorem 2 we may conclude that $F(x) = f(x) \cdot g(x)$ is continuous at $x = 2$.

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Written Assignment

Solve Problems 1-10, omit 9, pages 94-95.

Review Assignment

Solve Problems 1-70, multiples of 5, pages 96-99.

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Give the prepared work for Lessons 10 and 11 to the supervisor who will complete the mailing procedure.

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When the student has received the evaluated Written Assignments for Lessons 10 and 11 and has reviewed the work, he may request the supervisor to administer the hour examination.

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Lesson 12

INCREASING AND DECREASING FUNCTIONS AND RELATED RATES

Reference Material

Thomas, Chapter 3, Sections 1 and 2

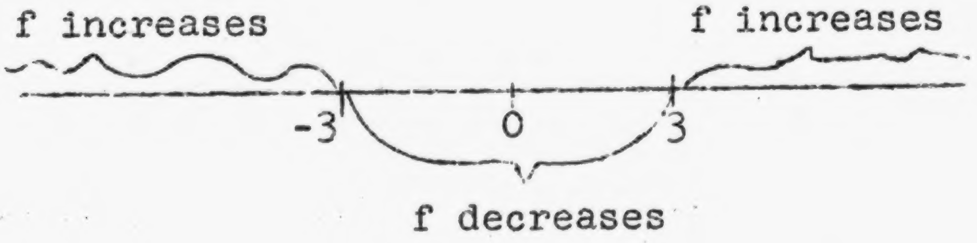
Supplementary Explanation

1. In Section 1 of this chapter, we begin with another application of the derivative. The essential result of this section is given as follows. If a function f is continuous for all $a \leq x \leq b$ and if $f'(x) > 0$ for all $a < x < b$, then f is increasing for all $a \leq x \leq b$. On the other hand, if $f'(x) < 0$ for all $a < x < b$, then f is decreasing for all $a \leq x \leq b$. This result is proved in Section 10 of this chapter.

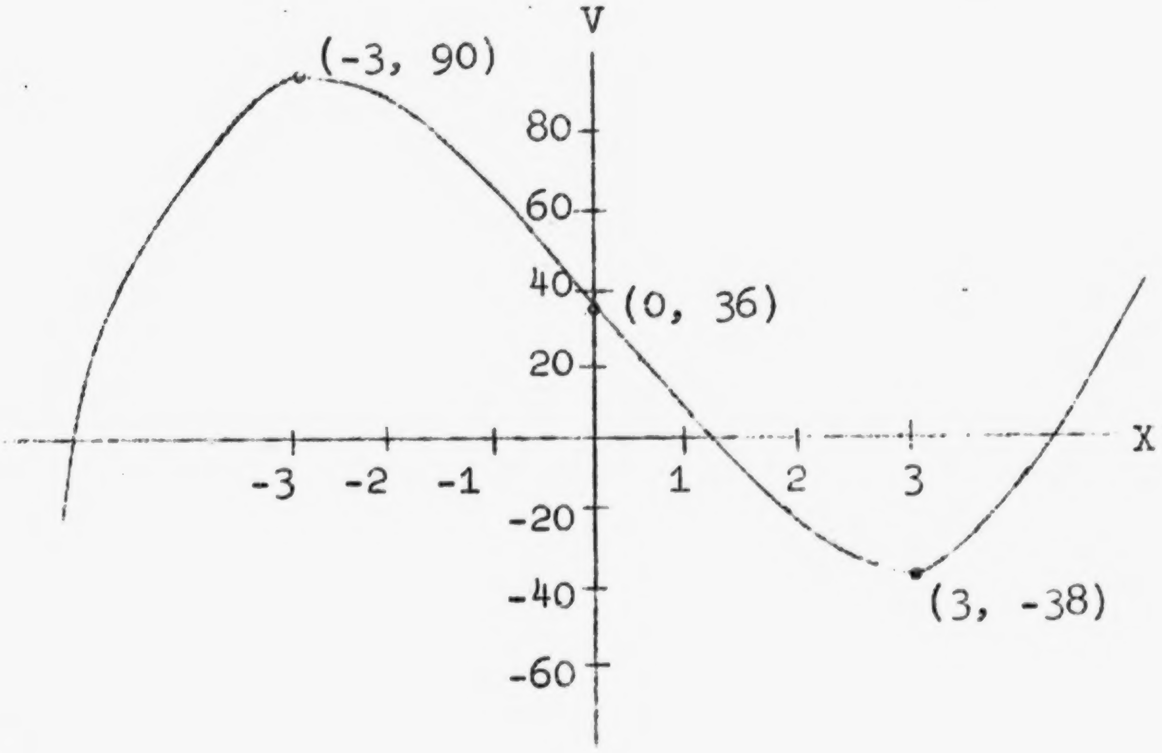
   There are many applications of this result, in particular, it is extremely useful in the problem of graphing a function.

Example 1: Problem 4, page 102 in the textbook

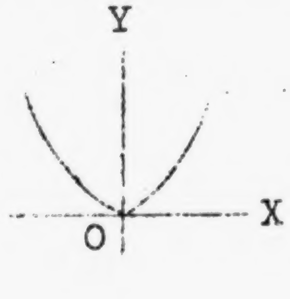
Solution: Since $y = f(x) = x^3 - 27x + 36$ we have $f'(x) = 3x^2 - 27$. Now $f'(x) = 0$ when $3x^2 - 27 = 0$, or when $x = \pm 3$. When $x = 3$, $y = 18$ and when $x = -3$, $y = -92$. Thus the points (3, -38) and (-3, 90) will mark the transition in the signs of the slopes. (They also could be "high" or "low" points.) Now f will be increasing if $f'(x) = 3(x^2 - 9) > 0$ or when $x^2 - 9 = (x - 3)(x + 3) > 0$. Now $(x - 3)(x + 3) > 0$ when either $x - 3 > 0$ and $x + 3 > 0$ or when $x - 3 < 0$ and $x + 3 < 0$. In the first case we have $x > 3$ and $x > -3$ and these imply that $\boxed{x > 3}$. In the second case we have $x < 3$ and $x < -3$ and these imply that $\boxed{x < -3}$. Hence f will increase when $x > 3$ and when $x < -3$. Consequently f will decrease when $-3 < x < 3$.



From this diagram we may conclude that when $x = -3$, f reaches a "high" point and when $x = 3$, f reaches a "low" point. We are almost ready to sketch the graph of f. Note that when $x = 0$, $y = 36$. Plotting the points (0, 36), (3, -38) and (-3, 90) and noting the above results yields the following graph.

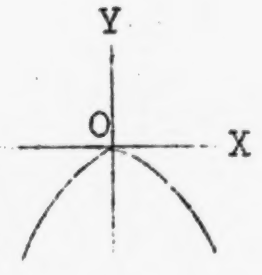


Examples 2-4: Consider the functions $f(x) = x^2$, $f(x) = -x^2$, $f(x) = x^3$, and $f(x) = -x^3$, whose graphs are illustrated in Figures 1 through 4.



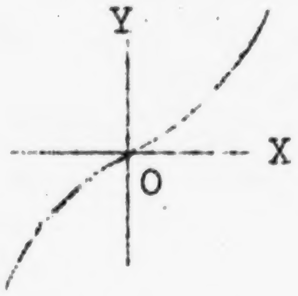
$f(x) = x^2$

Figure 1



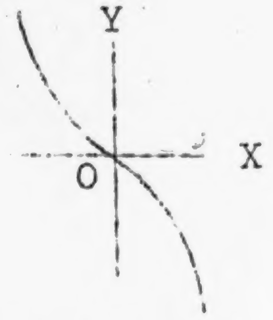
$f(x) = -x^2$

Figure 2



$f(x) = x^3$

Figure 3



$f(x) = -x^3$

Figure 4

For each of these functions, $f'(x) = 0$ if and only if $x = 0$. However the behavior of these four functions for $x \neq 0$ is quite different. For the first function, $f(x) = x^2$, we have $f'(x) = 2x$; hence $f'(c) < 0$ for $c < 0$ and $f'(c) > 0$ for $c > 0$. Thus $f(x) = x^2$ is decreasing in $(-\infty, 0]$ and increasing in $[0, \infty)$, and has a minimum value at $x = 0$. The function $f(x) = -x^2$ has derivative $f'(x) = -2x$; therefore $f'(c) > 0$ for $c < 0$ and $f'(c) < 0$ for $c > 0$. Thus $f(x) = -x^2$ is increasing in the interval $(-\infty, 0]$ and decreasing in the interval $[0, \infty)$, and it has a maximum value at $x = 0$.

Consider the third function, $f(x) = x^3$. Its derivative is $3x^2$, so $f'(c) > 0$ for all $c \neq 0$; hence $f(x) = x^3$ is increasing in $(-\infty, \infty)$. This function therefore has no maximum value and no minimum value in any open interval. Of course, in any closed interval $[a, b]$, where $a < b$, the function $f(x) = x^3$ assumes a minimum value at $x = a$ and a maximum value at $x = b$. Similarly the fourth function, $f(x) = -x^3$, has derivative $f'(x) = -3x^2$ so, since $f'(c) < 0$ for all $c \neq 0$, this function is decreasing in $(-\infty, \infty)$. Again $f(x) = -x^3$ has neither a maximum value nor a minimum value in any open interval, while in a closed interval $[a, b]$ (where $a < b$) it assumes its maximum value at $x = a$ and its minimum value at $x = b$.

Observe that each of the four functions has a horizontal tangent at $x = 0$, whereas only the first two functions have extreme values at $x = 0$. This shows that $f'(c) = 0$ does not imply, in general, that f has an extreme value at $x = c$.

2. Now we move to a consideration of Section 2. We present the solutions to the following problems to supplement those given in the texts. Each of these illustrations should be studied very carefully before proceeding to the assignment.

Example 1: Problem 6, page 106 in the textbook

Solution: Let V and r represent the volume and radius, respectively of the spherical balloon. Thus, we are given $\frac{dV}{dt} = 100$ ft.^{3}/min. and wish to find $\frac{dr}{dt}$ when r = 3 ft. Now we know the volume of a sphere is given by

$$V = \frac{4}{3}\pi r^3$$

and hence differentiating with respect to t yields

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

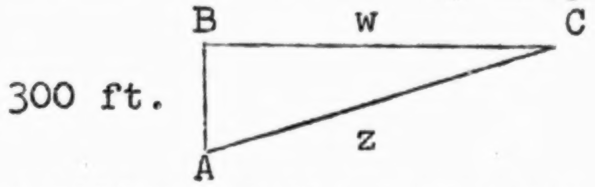
Now when r = 3 ft., $\frac{dV}{dt} = 100$ ft.^{3}/min. and so we have

$$100 \frac{(\text{ft.})^3}{\text{min.}} = 4\pi \cdot 9(\text{ft.})^2 \frac{dr}{dt} \text{ or}$$

$$\frac{dr}{dt} = \frac{25 \text{ ft.}}{9\pi \text{ min.}}$$

Example 2: Problem 14, page 107

Solution: Consider the following diagram.



Let A be the position of the boy and B a position of the kite (given as 300 ft. above A). Let C be the position of the kite at a later time. Further, if $|BC| = w$ and $|AC| = z$, then it follows that

(1) $$z^2 = w^2 + (300)^2$$

therefore,

$$2z\frac{dz}{dt} = 2w\frac{dw}{dt} \text{ or}$$

(2) $$z\frac{dz}{dt} = w\frac{dw}{dt}.$$

Now when z = 500 ft., $w = \sqrt{(500)^2 - (300)^2} = 400$ (this follows from (1)), and $\frac{dw}{dt} = 25$. This information along with (2) yields

$$500 \text{ ft. } \frac{dz}{dt} = 400 \text{ ft. } 25 \text{ ft./sec. or}$$

$$\frac{dz}{dt} = 20 \text{ ft./sec.}$$

Example 3: We next consider the following problem. At noon a vessel is sailing due north at the uniform rate of 15 miles per hour. Another vessel, 30 miles due north of the first, is sailing due east at the uniform rate of 20 miles per hour. At what rate is the distance between the vessels changing at the end of one hour?

Solution: In Figure 5, let A and B designate the positions of the two vessels at noon, with $|AB| = 30$. Let S_1 and S_2 be the corresponding positions of the vessels at any later time t. Let $x = |AS_1|$ be the distance in miles that the first vessel has traveled in t hours, and let $y = |BS_2|$ be the distance in miles that the second vessel has traveled in t hours. Then $z = |S_1S_2|$ is the distance in miles between the vessels at the time. We are given that $\frac{dx}{dt} = 15$, $\frac{dy}{dt} = 20$. Now at any time t,

(1) $z^2 = (30 - x)^2 + y^2$, from right triangle S_1BS_2. Therefore, $2z\frac{dz}{dt} = 2(30 - x)(-1)\frac{dx}{dt} + 2y\frac{dy}{dt}$, or

(2) $z\frac{dz}{dt} = -(30 - x)\frac{dx}{dt} + y\frac{dy}{dt}$.

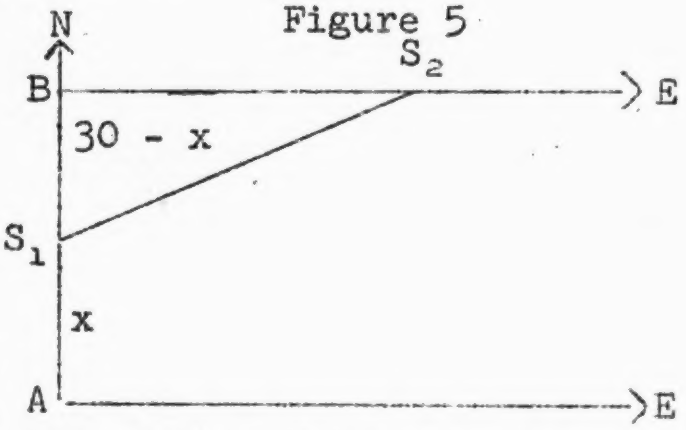


Figure 5

This gives a relation between the rates of change of z, x, and y at any time t. Now let $t = 1$; then $x = 15$, $y = 20$, and, by (1), $z = 25$. Substituting $z = 25$, $x = 15$, $y = 20$, $\frac{dx}{dt} = 15$, and $\frac{dy}{dt} = 20$ in (2) we find that $\frac{dz}{dt} = 7$. Hence at the end of one hour the vessels are separating at the rate of 7 miles per hour.

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Written Assignment

Solve Problems 1, 3, 5, page 102 in the textbook. Solve Problems 3, 7, 9, 11, 13, 15, pages 106-107 in the textbook.

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Give the prepared work for Lesson 12 to the supervisor who will complete the mailing procedure.

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Lesson 13

THE SECOND DERIVATIVE AND CURVE PLOTTING

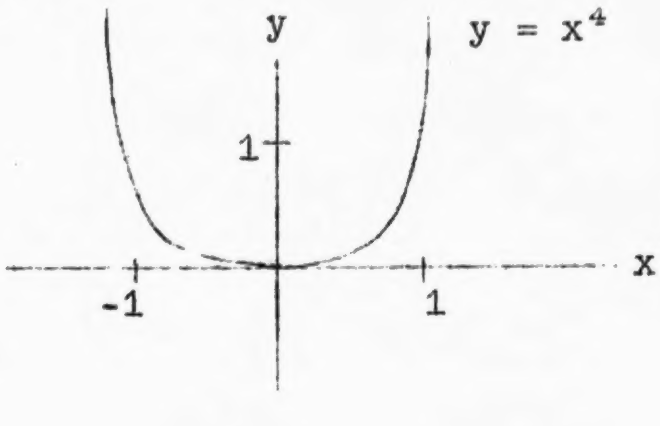
Reference Material

Thomas, Chapter 3, Sections 3 and 4

Supplementary Explanation

1. In Section 1 we saw how one can determine when a function f is increasing ($f'(x) > 0$) and when it is decreasing ($f'(x) < 0$). Now in the beginning of this Section 3, the author emphasizes the fact that the points, which separate the regions of rise and fall of the curve need not occur when $f'(x) = 0$ and further the fact that $f'(x) = 0$ for some x does not necessarily imply that we have high or low points at these values of x. In other words, it is true that if $f(d)$ is a maximum or minimum value of f in $[a, b]$, then either $f'(d) = 0$ or $f'(d)$ does not exist. However, the converse of this result is not true in general. Study the examples on pages 107 and 108.

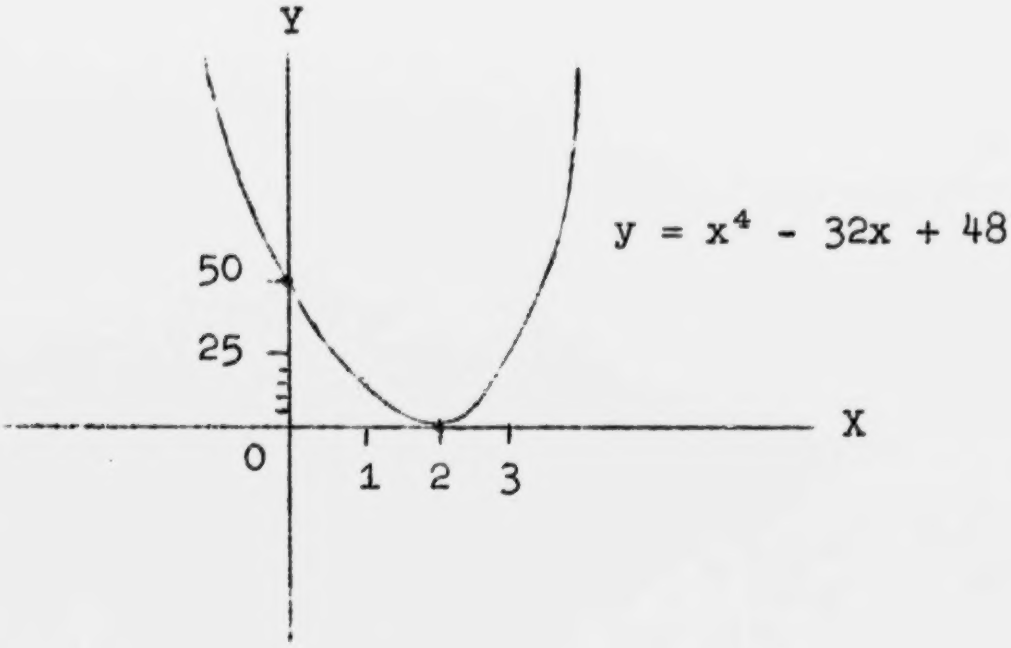
2. The notion of concavity is next discussed and we have the result that the graph of f is concave upward if $f''(x) > 0$ and concave downward when $f''(x) < 0$. A point at which the curve changes from concave upward to concave downward or vice versa is called a point of inflection, i.e., where $f''(x)$ changes from a positive value to a negative value or again vice versa. Obviously, such a change may occur where $f''(x) = 0$ but in addition, the change in sign of the second derivative may occur where $f''(x)$ fails to exist as the example on page 109 illustrates. Once again we should emphasize that these conditions for determining points of inflection are necessary but not sufficient. For example, let $f(x) = x^4$ then $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Now $f''(x) = 0$ when $12x^2 = 0$, i.e., when $x = 0$. However the point $(0, f(0)) = (0, 0)$ is not a point of inflection since $f''(x)$ is positive for all $x \neq 0$ and consequently, the graph of $y = f(x)$ will always be concave upward. [Note that $f'(x) = 0$ when $x = 0$ and $f'(x) > 0$ when $x > 0$, while $f'(x) < 0$ when $x < 0$ and so $(0, 0)$ is the low point of $f(x) = x^4$.] We have graphically:



3. A procedure which might be used to sketch the graph of a function is outlined in Section 4. Study the examples given on pages 110-113 very carefully. We shall present several more as supplementary examples.

Example 1: Problem 11, page 114

Solution: Since $y = f(x) = x^4 - 32x + 48$ we have $f'(x) = 4x^3 - 32$ and $f''(x) = 12x^2$. Immediately we conclude that there are no points of inflection and the graph will always be concave upward since $f''(x) > 0$ for all $x \neq 0$. Now $\overline{f'(x) < 0}$ when $\overline{4(x^3 - 8)} < 0$ or when $x < 2$. Hence the graph will be decreasing when $x < 2$, increasing when $x > 2$ and thus reach a low point when $x = \overline{2}$, i.e. at the point $(2, f(2)) = (\overline{2, 0})$. Now when $x = 0$, $y = 48$ and so our graph takes the following form:



Example 2: Problem 20, page 114

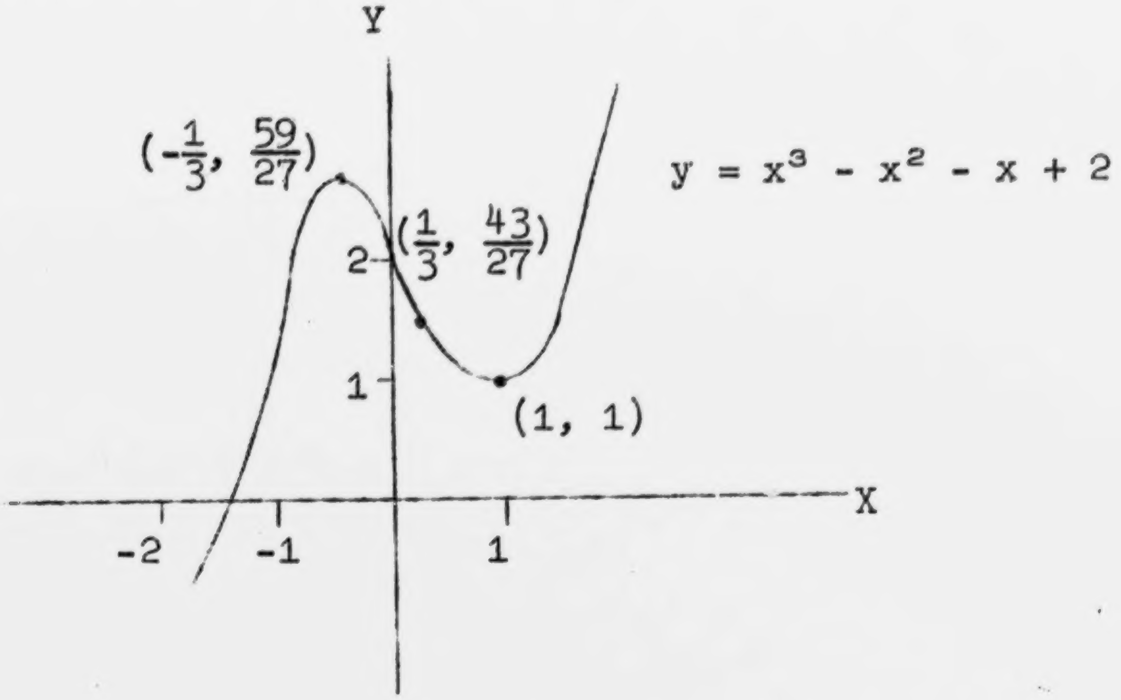
Solution: Let x be any positive real number. We wish to find the minimum value of $f(x) = x + \frac{1}{x}$. The desired result follows from Example 2 found on page 112.

Example 3:

Consider the following problem. Discuss and sketch the graph of $y = x^3 - x^2 - x + 2$.

Solution: Now $y = f(x) = x^3 - x^2 - x + 2$ and therefore $f'(x) = 3x^2 - 2x - 1$ and $f''(x) = 6x - 2$. Thus $f'(x) = 0$ when $3x^2 - 2x - 1 = (3x + 1)(x - 1) = 0$ or when $x = -\frac{1}{3}$

$x = 1$. Hence we have possible maximum or minimum points at $(-\frac{1}{3}, f(-\frac{1}{3})) = (-\frac{1}{3}, \frac{59}{27})$ and $(1, 1)$. Further $f''(x) = 0$, when $x = \frac{1}{3}$ and therefore $(\frac{1}{3}, \frac{43}{27})$ is a possible point of inflection. The curve will be increasing when $f'(x) > 0$ or when $(3x + 1)(x - 1) > 0$ which implies that $x > 1$ or $x < -\frac{1}{3}$. Consequently, the curve will decrease for all $-\frac{1}{3} < x < 1$. Since $f''(x) > 0$ when $x > \frac{1}{3}$ so the curve is concave upward when $x > \frac{1}{3}$ and downward when $x < \frac{1}{3}$, we may conclude that the point $(\frac{1}{3}, \frac{43}{27})$ is a point of inflection and that we have a minimum at $(1, 1)$ and a maximum at $(-\frac{1}{3}, \frac{59}{27})$. Now when $x = 0$, $y = 2$. Thus collecting all of the above information we have the following sketch.



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Written Assignment

Solve Problems 3, 5, 10, 12, 13, 17, and 21, pages 113-115.

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Give the prepared work for Lesson 13 to the supervisor who will complete the mailing directions.

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Lesson 14

MAXIMA AND MINIMA THEORY AND APPLICATIONS

Reference Material

Thomas, Chapter 3, Sections 5 and 6

Mathematics XVxAP, page 55

Supplementary Explanation

1. In Section 5, note well the difference between relative and absolute extrema. Study carefully the theorem and its proof found on page 116 and the related Remark found at the bottom of the page.

   Before proceeding we shall state rigorously a definition which will introduce a convenient notation for intervals. Definition: Given the real numbers a and b with $a < b$. The open interval determined by a and b is the set of all real numbers x such that $a < x < b$ and it is designated by (a, b) while the closed interval is the set of real numbers x such that $a \leq x \leq b$ and it is designated by $[a, b]$. What will $[a, b)$ and $(a, b]$ represent? Now (a, ∞) represents the set of all real numbers x greater than a. What will $(-\infty, a)$, $(-\infty, a]$ and $(-\infty, \infty)$ represent? These notations will be used throughout the remainder of the course.

   Endpoint extrema may be identified in the following way. Let $P(c, f(c))$ be an endpoint of the graph of f, and suppose that d is a critical number in the domain of f such that f has no critical number in the open interval (c, d). Then P is a maximum point of the graph of f (and $f(c)$ is a maximum value of f) if $f(c) > f(d)$, and P is a minimum point of the graph of f (and $f(c)$ is a minimum value of f) if $f(c) < f(d)$.

   Consider the following example. Let p, q, n be integers and $f(x) = x^{\frac{p}{q}}(x - 1)^n$, $0 < p < q$, $n > 2$. The fraction $\frac{p}{q}$ is assumed to be in lowest form. Find the extrema of f for all choices of p, q, and n as even or odd integers.

   Solution: Consider the function f defined by

   $$f(x) = x^{\frac{p}{q}}(x - 1)^n, \quad 0 < p < q, \; n \geq 2,$$

   where p, q, n are integers. The domain of f is $(0, \infty)$, and clearly f is continuous in this domain. Also

   $$\begin{aligned} f'(x) &= \frac{p}{q}x^{\frac{p}{q}-1}(x - 1)^n + nx^{\frac{p}{q}}(x - 1)^{n-1} \\ &= \frac{x^{\frac{p}{q}}}{x}(x - 1)^{n-1}\left[\frac{p}{q}(x - 1) + nx\right]. \end{aligned}$$

   Certainly $f'(x)$ exists for all $x > 0$, whereas $f'(0)$ does not exist. Hence $x = 0$ is a critical number of f. The remaining critical numbers are obtained by setting $f'(x) = 0$ and solving for x; they are seen to be $x = 1$ and

$x = \frac{p}{p + qn}$. Since $(0, f(0))$ is an endpoint of the graph of f and since 1 and $\frac{p}{p + qn}$ are the only other critical numbers of f, we may identify the extrema of f by considering the open intervals $(0, 1)$ and $\left(\frac{p}{p + qn}, 2\right)$; the former contains $\frac{p}{p + qn}$ and the latter contains 1. Now $f(0) = 0$, $f(1) = 0$, $f(2) = 2^{\frac{p}{q}} > 0$, and, as is readily verified, $f\left(\frac{p}{p + qn}\right) = \left(\frac{p}{p + qn}\right)^{\frac{p}{q}}\left(\frac{qn}{p + qn}\right)^{n}(-1)^{n}$; hence $f\left(\frac{p}{p + qn}\right) > 0$ if n is even, while $f\left(\frac{p}{p + qn}\right) < 0$ if n is odd.

a. If n is even then $f\left(\frac{p}{p + qn}\right) > f(0)$ and $f\left(\frac{p}{p + qn}\right) > f(1)$, so $f\left(\frac{p}{p + qn}\right)$ is a maximum value of f. Since f has no critical number between 0 and $\frac{p}{p + qn}$, $f(0)$ is an endpoint minimum value of f. Since $f(1) < f\left(\frac{p}{p + qn}\right)$ and $f(1) < f(2)$, $f(1)$ is also a minimum value of f.

b. If n is odd then $f\left(\frac{p}{p + qn}\right) < f(0)$ and $f\left(\frac{p}{p + qn}\right) < f(1)$; hence $f\left(\frac{p}{p + qn}\right)$ is a minimum value of f. Thus $f(0)$ is an endpoint maximum value of f. Since $f(1) > f\left(\frac{p}{p + qn}\right)$ and $f(1) < f(2)$, $f(1)$ is not an extremum of f.

The two cases are illustrated by Figures a and b.

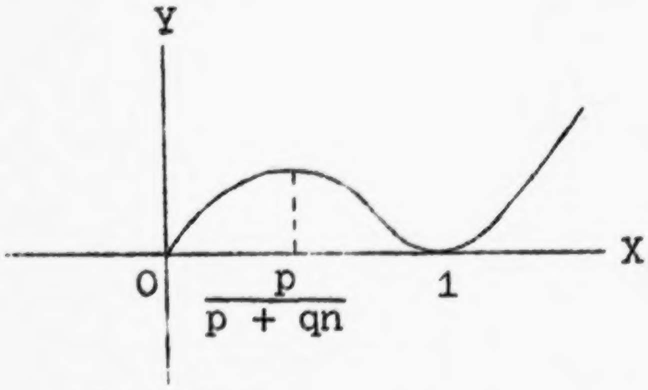


Figure a

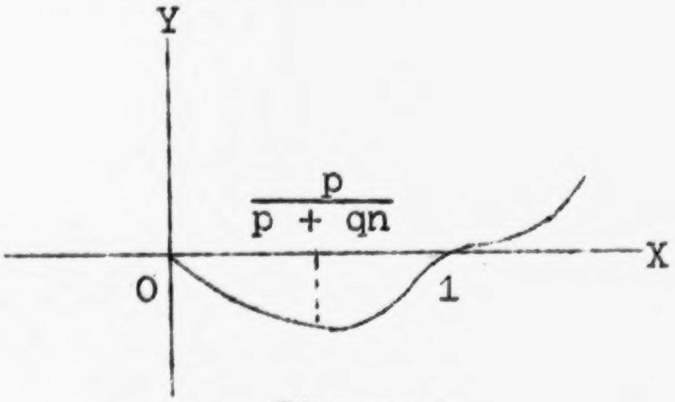
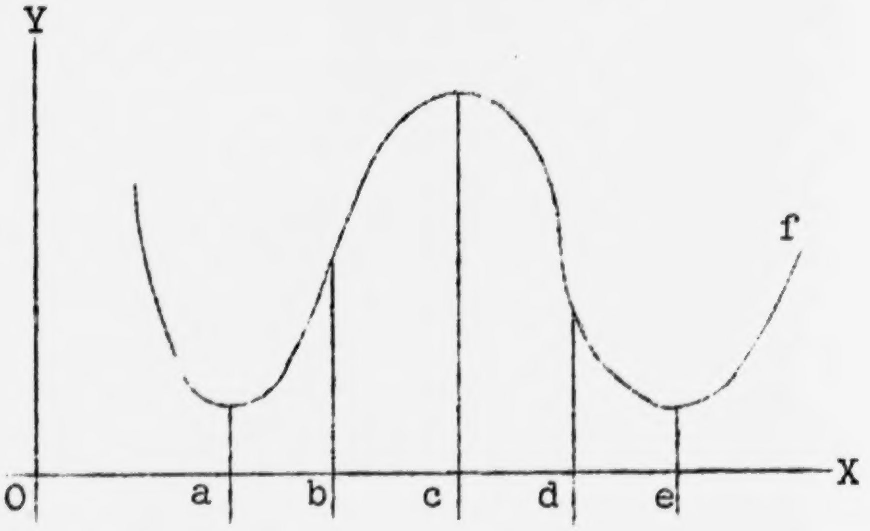


Figure b

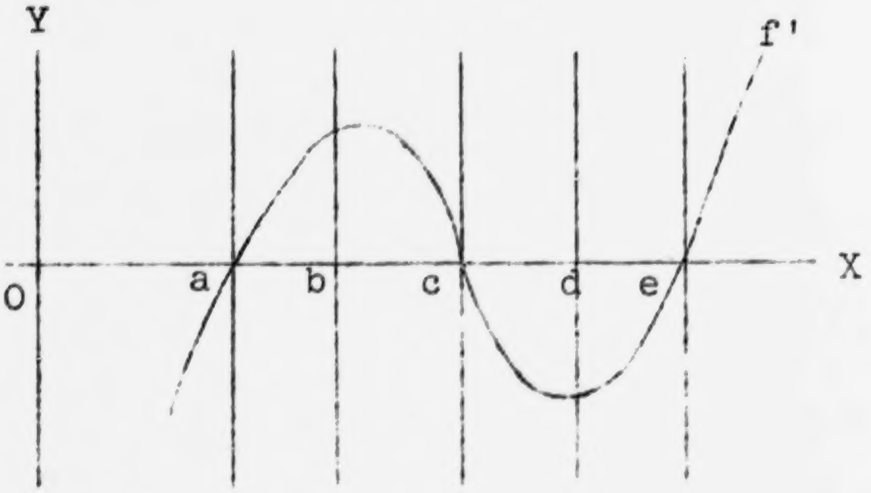
2. Section 6: Before proceeding with this explanation study the illustrative examples given in the text. Note that in each of these problems one is searching for either an absolute maximum or minimum. Some rigorous justification of your conclusions must be presented. Usually a consideration of your answer in relation to the given conditions and the use of the second derivative test for extrema will be sufficient. The author has provided on page 126 an excellent summary of procedure which might be followed to gain a solution to the "applied problems". This procedure contains a summary of results which you should know (memorize if necessary) so that it may be used at any time without hesitation. Notice that if the "second derivative" test (a) fails then you must use the "first derivative" test (b).

   One might find it rather easy to remember the various tests if one considers the relationships between the graph of f and the graphs of f', f" and f"'. Consider the figures on the following page.

Minimum at $(a, f(a))$
and $(e, f(e))$;
Maximum at $(c, f(c))$;
Points of inflection at
$(b, f(b))$ and $(d, f(d))$.



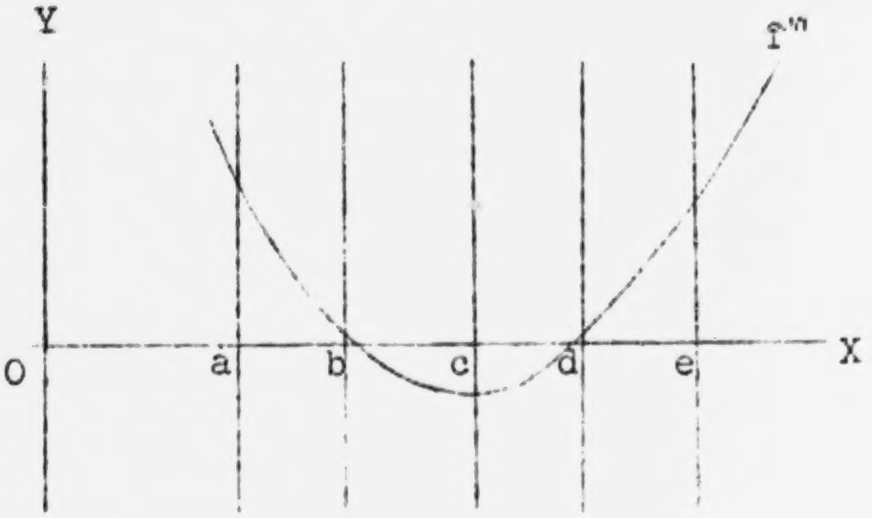
$f'(a) = f'(c) = f'(e) = 0$. Notice that the graph of f' has a maximum at $(b, f(b))$ and a minimum at $(d, f(d))$.



$f''(a) > 0$, $f''(e) > 0$;

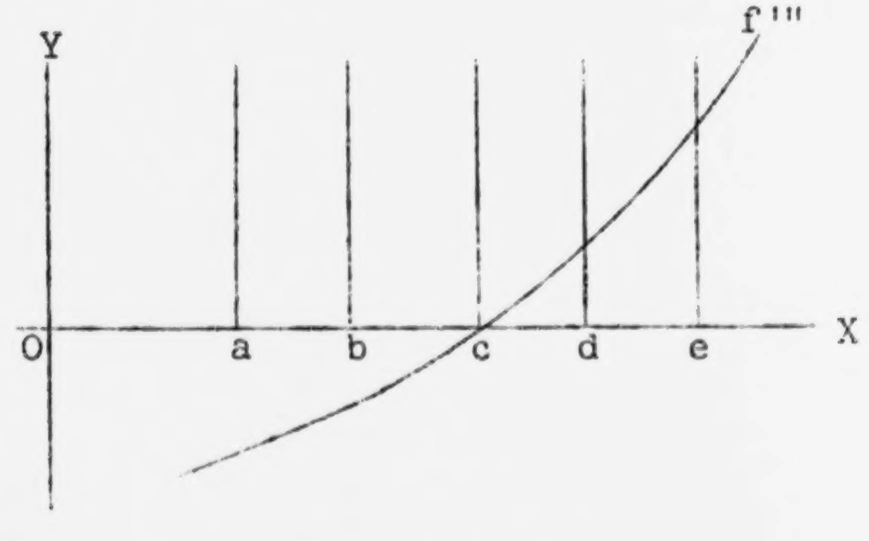
$f''(c) < 0$;

$f''(b) = f''(d) = 0$.



$f'''(b) < 0$.

$f'''(d) > 0$.



In addition to the first and second derivative tests for extrema one can also draw from the graphs various results for determining (i) when f is increasing or decreasing (check the sign first derivative from a to c and then from c to e), (ii) when f is concave upward and downward and (check the second derivative graph), (iii) the point(s) of inflection. Note that f has points of inflection at $x = b$ and $x = d$ and at these points $f''(x) = 0$ and $f'''(x)$ is either positive or negative. Hence if at the values of x for which $f''(x) = 0$, $f'''(x)$ is > 0 or < 0 then these x values will determine points of inflection. However, this test fails when at these points $f'''(x) = 0$. In this case what test would you use for points of inflection? (See the graph of $f''(x)$ and note its sign change for $x < b$, $x = b$ and $x > b$. Do the same for $x = d$.) For example, if $f(x) = x^4$ then $f'(x) = 4x^3$, $f''(x) = 12x^2$, $f'''(x) = 24x$. Now $f''(0) = 0$ so $x = 0$ could yield a point of inflection. However the third derivative test fails since $f'''(0) = 0$. Do we have a point of inflection at $x = 0$?

3. Now regarding the applications there is naturally no set procedure for every problem which will enable one to set up an analytic equation which describes the physical situation. Practice is the best teacher of technique. Oftentimes one needs to use results from other branches of mathematics e.g. geometry (see Appendix pages 565-568).

Consider the following Problem 19, page 128.

Solution:

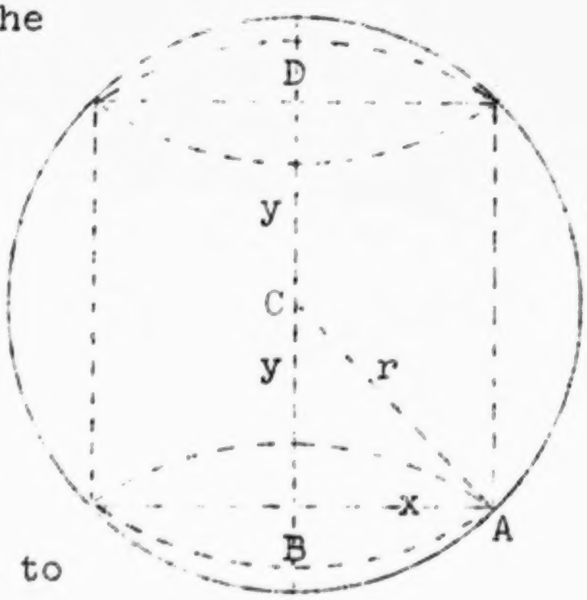
In the notation of the figure, the volume of the inscribed cylinder is $V = 2\pi x^2 y$, where x and y must satisfy

(a) $x^2 + y^2 = r^2$.

Hence

(b) $V = 2\pi(r^2 y - y^3)$.

[Here $x = |AB|$, $y = |BC| = |CD|$, $r = |AC|$.]



Differentiating (b) with respect to y yields

(c) $\frac{dV}{dy} = 2\pi(r^2 - 3y^2)$.

If $\frac{dV}{dy} = 0$ then $r^2 - 3y^2 = 0$ so, since $y \geq 0$, $y = \frac{r\sqrt{3}}{3}$.

Moreover, we see from (c) that $\frac{d^2V}{dy^2} = 2\pi(-6y)$.

Hence if $y = \frac{r\sqrt{3}}{3}$ then $\frac{d^2V}{dy^2} < 0$, so V has a maximum value at $y = \frac{r\sqrt{3}}{3}$. When $y = \frac{r\sqrt{3}}{3}$ we see from (a) that $x = \frac{r\sqrt{6}}{3}$. Hence the cylinder of maximum volume inscribed in a sphere of radius r has altitude $\frac{2r\sqrt{3}}{3}$ and base of radius $\frac{r\sqrt{6}}{3}$. Since $V = 2\pi x^2 y$ our maximum volume is

$$V = 2\pi\left(\frac{r\sqrt{6}}{3}\right)^2\left(\frac{r\sqrt{3}}{3}\right) = \frac{4}{9}\pi r^3\sqrt{3} \text{ cubic units.}$$

Consider the following problem:

Express the number 12 as the sum of two positive numbers in such a way that the product of one by the square of the other is as large as possible.

Solution: Let $x > 0$ and $y > 0$ be the two numbers sought. Then we have

(a) $\quad x + y = 12$
and
(b) $\quad x^2y = c$.

We wish to maximize c. From (a) and (b) we have

$$c = x^2(12 - x) = 12x^2 - x^3$$

and so

$$\frac{dc}{dx} = 24x - 3x^2 = 3x(8 - x).$$

Now $\frac{dc}{dx} = 0$ when $3x(8 - x) = 0$ or

when $x = 0$ and $x = 8$. We do not accept $x = 0$ since the problem requires x to be positive.

Now $\frac{d^2c}{dx^2} = 24 - 6x$ and when $x = 8$, $\frac{d^2c}{dx^2} = 24 - 48 < 0$
$\therefore$ we have a maximum when $x = 8$. Since $x + y = 12$, it follows that $y = 4$. The maximum product $c = 64 \cdot 4 = \underline{256}$.

Consider the following problem:

A solid is formed by cutting hemispherical cavities from the ends of a right circular cylinder, the bases of the hemispheres coinciding with the ends of the cylinder. If the total area of the solid is a specified constant, find the ratio of height to radius of base for the cylinder so as to give the solid a maximum volume.

Solution: Let the height of the cylinder be h and the radius of the sphere and cylinder be r. We have for the surface area of the solid:

$A = 2\pi rh + 4\pi r^2$, A is constant and its volume is given by

$V = \pi r^2 h - \frac{4}{3}\pi r^3$.

We wish to maximize V. Assuming r is the independent variable we have

(1) $\frac{dA}{dr} = 2\pi h + 2\pi r\frac{dh}{dr} + 8\pi r = 0$, since A is constant

(2) $\frac{dV}{dr} = 2\pi rh + \pi r^2\frac{dh}{dr} - 4\pi r^2 = 0$, since we want to maximize V. Now from (1) we have $\frac{dh}{dr} = -\left(\frac{h + 4r}{r}\right)$, substituting in to (2) yields $2\pi rh + \pi r^2\left(\frac{-h - 4r}{r}\right) - 4\pi r^2 = 0$ which simplifies to $r = 0$ (not acceptable) or $h - 8r = 0$. From this latter equation we have $\frac{h}{r} = 8$ and this will yield the the maximum V since $\frac{d^2V}{dr^2} < 0$ when $h = 8r$.

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Written Assignment

Solve Problems 1, 4, 7, 11, 20, 23, 24, and 26, pages 127-129.

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Give the prepared work for Lesson 14 to the supervisor who will complete the mailing procedure.

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Lesson 15

ROLLE'S THEOREM AND THE MEAN VALUE THEOREM

Reference Material

Thomas, Chapter 3, Sections 7 and 8

Supplementary Explanation

1. Section 7: There are many applications of Rolle's Theorem one of which is given in Remark 2. Another application of the theorem is given in the proof of the Mean Value Theorem which is found in the next section. We shall add a few more examples.

Example 1: Verify Rolle's Theorem for the function described by $f(x) = x^3 - 9x$ in the interval $0 \leq x \leq 3$.

Solution:

Note that $f(x) = x^3 - 9x$ is continuous and differentiable for all x such that $-\infty < x < +\infty$. Why? Further $f(0) = f(3) = 0$, consequently the hypotheses of Rolle's Theorem are satisfied. We may thus conclude that there is at least one number c between 0 and 3 such that $f'(c) = 0$. In fact since $f'(x) = 3x^2 - 9$,

we see that $f'(\pm\sqrt{3}) = 0$ so $c = \sqrt{3}$ or $c = -\sqrt{3}$.

Example 2:

The function

$$f(x) = \begin{cases} 0 & \text{when } x - 1 \\ x - 3 & \text{when } 1 < x < 4 \\ 0 & \text{when } x = 4 \end{cases}$$

is zero when $x = 1$ and $x = 4$. Is Rolle's Theorem applicable to this function?

Solution: No Rolle's Theorem is not applicable since $f(x)$ is not continuous for all x in $1 \leq x \leq 4$. Why? Sketch this function.

2. Proceed with the following discussion only after you have studied Section 8 very carefully. Note the precise statement of the Mean Value Theorem and how Rolle's Theorem is used in its proof. These two theorems which seem to be rather "obvious" in their geometrical interpretation lead to many significant results in mathematics (see for instance Corollaries 1 and 2 on page 136). You will be responsible for the precise statements of these theorems as well as their geometrical interpretation. We will provide some additional applications of the Mean Value Theorem.

Example 1: Proof of Corollary 2, page 136.

Consider the function $F(x) = F_1(x) - F_2(x)$. By hypothesis

$F'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$ for $a < x < b$.

Hence Corollary 1 applies and asserts that $F(x)$ is constant

for $a < x < b$. This proves Corollary 2.

Example 2: Find upper and lower bounds for $\sqrt{99}$.

Solution: We know $\sqrt{100} = 10$ and under proper conditions on a function f we have from the Mean Value Theorem (MVT)

$\frac{f(b) - f(a)}{b - a} = f'(c)$ or $f(b) - f(a) = (b - a)\, f'(c)$ where $a < c < b$.

Now let $f(x) = \sqrt{x}$ and choose the interval $99 < x < 100$.

Now since $f'(x) = \frac{1}{2\sqrt{x}}$ exists in this interval the MVT applies and hence we know there exists a number c such that $99 < c < 100$ such that

$f(100) - f(99) = (100 - 99)\frac{1}{2\sqrt{c}}$ or

(1) $\qquad 10 - \sqrt{99} = \frac{1}{2\sqrt{c}}$.

Since $99 < c < 100$ we may say that $\sqrt{99} < \sqrt{c} < 10$ or $9 < \sqrt{c} < 10$ (9 is chosen for simplicity we could have chosen a larger number such at 9.5 and have a sharper result). Then $\frac{1}{9} > \frac{1}{\sqrt{c}} > \frac{1}{10}$ and $\frac{1}{18} > \frac{1}{2\sqrt{c}} > \frac{1}{20}$. Thus we have from this result and (1):

$\frac{1}{18} > 10 - \sqrt{99} > \frac{1}{20}$ or

$\frac{1}{18} - 10 > -\sqrt{99} > \frac{1}{20} - 10$ or

$10 - \frac{1}{18} < \sqrt{99} < 10 - \frac{1}{20}$ and finally

$9.945 < \sqrt{99} < 9.95$.

Example 3: Given $f(x) = \frac{1}{x}$ and $a = 1$, $b = 3$. Can one apply the MVT? If so, find c.

Solution: The MVT is obviously applicable since f is continuous and $f'(x) = -\frac{1}{x^2}$ exists in the interval $1 \leq x \leq 3$. Thus we know there exists a number c in the interval $1 < c < 3$ such that

$f(3) - f(1) = (3 - 1)(-\frac{1}{c^2})$ or

$\frac{1}{3} - 1 = 2\,(-\frac{1}{c^2})$ which implies

$-\frac{2}{6} = -\frac{1}{c^2}$ and

$\frac{1}{3} = \frac{1}{c^2}$ or $c^2 = 3$ and thus $c = \pm\sqrt{3}$.

Since $-\sqrt{3}$ does not lie in the interval we accept only the value $c = \sqrt{3}$.

3. The following are final remarks concerning Chapter 3. Sections 9 and 10 will not be covered in this course. However you should read them at least once since they do contain certain generalizations and applications which may be useful to you in future courses in mathematics (they are considered in advanced courses). Section 10 in fact contains the proofs of many of the results stated and used earlier in this chapter (see Sections 3-1 through 3-6). We conclude this lesson with the proof of one of these results.

   Theorem: If $f'(x) < 0$ for $r \leq x \leq s$ then $f(x)$ is a decreasing function for the interval $r \leq x \leq s$.

   Proof: Let a and b, $b > a$, be any two points of the interval $r \leq x \leq s$. Since $f'(x)$ exists for $a \leq x \leq b$ the hypotheses of the MVT are satisfied and thus there exists a number c such that $r \leq a < c < b \leq s$ such that

   $$f(b) - f(a) = (b - a)\, f'(c). \tag{1}$$

   Now since $b > a$ we have $b - a$ is positive and $f'(c)$ is negative by hypothesis. Consequently, we may conclude from (1) that

   $$f(b) - f(a) < 0 \quad \text{which implies}$$

   that $f(b) < f(a)$ and hence $f(x)$ is decreasing in the interval $r \leq x \leq s$.

   The corresponding theorem for increasing functions is proved similarly.

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Written Assignment

Solve Problems 1, 2, and 3, page 132 in the textbook. Solve Problems 1, 3, 5, 6, and 7a, page 137. Solve Problems 10, 20, 30, 40, 50, 60, and 70, pages 145-149.

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Give the prepared work for Lesson 15 to the supervisor who will complete the mailing procedure.

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When the student has received the evaluated Written Assignment for Lesson 15 and has reviewed the work, he may request the supervisor to administer the hour examination.

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Lesson 16

SYMMETRY, EXTENT, ASYMPTOTES, INTERCEPTS

Reference Material

Thomas, Chapter 4, Section 1

Supplementary Explanation

1. This section contains certain results and definitions which will aid one in plotting "higher" plane curves. We shall elaborate on a few of these results. To emphasize the so called "cardinal principle" of analytic geometry (see page 151 of the textbook) we shall call it the Fundamental Principle of Analytic Geometry (FPAG). It states: if a point lies on a curve then the coordinates of the point satisfy an equation of the curve and conversely if an equation of a curve is satisfied by a point then the point lies on the curve. [Will the point (2, 1) lie on the curve described by $y = x^2 - 2x + 1$?] This principle will be used frequently in future work.

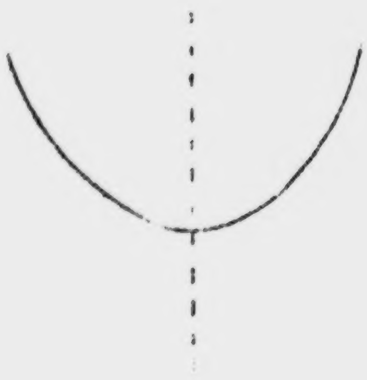
2. The notion of symmetry may be generalized as follows:

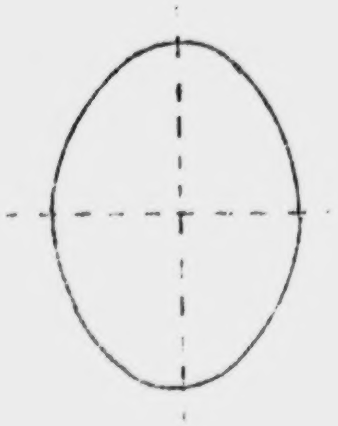
   a. Let L be a given line. Two points P and P' are said to be symmetric with respect to L if L is the perpendicular bisector of the line segment joining P and P'. A curve C is said to be symmetric with respect to L if for every point P on C there is another point P' on C such that P and P' are symmetric with respect to L. In this case, L is called an axis of symmetry of C. Determine a necessary and sufficient condition for the graph of an equation in x and y to be symmetric with respect to the line $y = -x$.

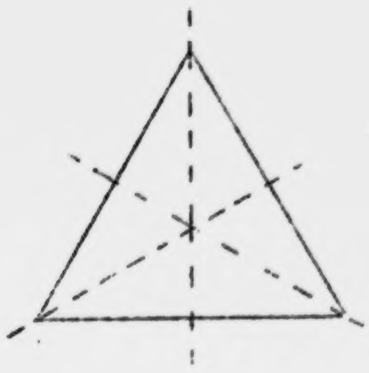
   b. Let Q be a given point. Two points P and P' are said to be symmetric with respect to Q if Q is the midpoint of the line segment joining P and P'. A curve C is said to be symmetric with respect to Q if for every P on C there is another point P' on C such that P and P' are symmetric with respect to Q. In this case Q is called a center of symmetry of C.

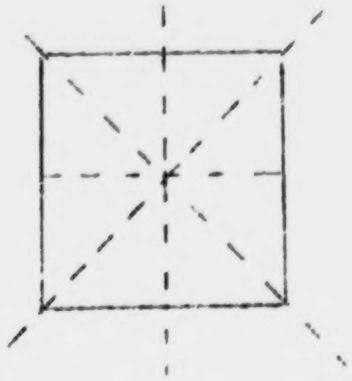
   The graph of an equation in x and y is symmetric with respect to the point Q(a, b) if and only if the equation is unchanged when x is replaced by $2a - x$ and y is replaced by $2b - y$. Show that the graph of the equation $x^2 - y^2 - 2x + 4y - 4 = 0$ is symmetric with respect to the point (1, 2).

c. The number of axes of symmetry which a curve possesses may be zero, N (where N is any positive integer), or infinite. That a curve may have no axis of symmetry is apparent. Furthermore if C is a circle, then any line passing through the center of C is an axis of symmetry of C. Moreover a parabola possesses just one axis of symmetry while an ellipse which is not a circle has exactly two axes of symmetry. Finally if N is any integer greater than 2 and if C is a regular N-sided polygon then a line L is an axis of symmetry of C if L passes through the center of C and a vertex of C. If N is odd, C has no other axes of symmetry, while if N is even, any perpendicular bisector of a side of C will be an axis of symmetry of C. Thus in any case, C has precisely N axes of symmetry.









4. We shall formally identify horizontal asymptotes by the following definition. The line $y = b$ is called a horizontal asymptote of the graph of a function f if and only if $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$.

For example, since

$$\lim_{x \to \infty} \frac{x^2 + x}{x^2 + 1} = \lim_{x \to \infty} \frac{1 + \frac{1}{x}}{1 + \frac{1}{x^2}} = 1$$

the line $y = 1$ is a horizontal asymptote. Note here that to evaluate the limit considered above we used a special case of the result that

(1) $$\lim_{x\to\infty} \frac{1}{x^p} = 0 \text{ if } p > 0.$$

For a rational function f it is easy to determine whether or not the graph of f has a horizontal asymptote. Suppose $f(x) = \frac{g(x)}{h(x)}$, where g and h are polynomial functions of x defined by

$$g(x) = a_0x^m + a_1x^{m-1} + \ldots + a_{m-1} x + a_m$$

and

$$h(x) = b_0x^n + b_1x^{n-1} + \ldots + b_{n-1} x + b_n,$$

with respective degrees m and n. We consider three cases:

(a) $\underline{m < n}$. In this case,

$$f(x) = \frac{a_0x^m + a_1x^{m-1} + \ldots + a_{m-1}x + a_m}{b_0x^n + b_1x^{n-1} + \ldots + b_{n-1}x + b_n}$$

$$= \frac{a_0x^{m-n} + a_1x^{m-1-n} + \ldots + a_{m-1}x^{1-n} + a_mx^{-n}}{b_0 + b_1x^{-1} + \ldots + b_{n-1}x^{-(n-1)} + b_nx^{-n}}.$$

Since $n - m > 0$, we may employ (1) together with the limit theorems to obtain

$$\operatorname{limit}_{x\to\pm\infty} f(x) = \frac{0 + 0 + \ldots + 0 + 0}{b_0 + 0 + \ldots + 0 + 0} = 0$$

Hence the graph of a rational function f has the line $y = 0$, (i.e., the x-axis) as a horizontal asymptote if the degree of the numerator of f is less than the degree of the denominator of f. For example, the graph of the equation $y = \frac{9x}{9 + x^2}$ has the line $y = 0$ as a horizontal asymptote.

(b) $\underline{m = n}$. Then

$$f(x) = \frac{a_0x^m + a_1x^{m-1} + \ldots + a_{m-1}x + a_m}{b_0x^n + b_1x^{n-1} + \ldots + b_{n-1}x + b_n}$$

$$= \frac{a_0 + a_1x^{-1} + \ldots + a_{m-1}x^{-(m-1)} + a_mx^{-m}}{b_0 + b_1x^{-1} + \ldots + b_{n-1}x^{-(n-1)} + b_nx^{-n}}.$$

Thus again by (1) and the limit theorems, we obtain

$$\lim_{x \to \pm\infty} f(x) = \frac{a_0 + 0 + \ldots + 0 + 0}{b_0 + 0 + \ldots + 0 + 0} = \frac{a_0}{b_0}.$$

Therefore the graph of a rational function f has the line $y = \frac{a_0}{b_0}$ as a horizontal asymptote if the degrees of the numerator and denominator of f are equal; here a_0 and b_0 are the coefficients of the highest power of x in the numerator and the denominator of f, respectively. For example, the graph of the equation $y = \frac{2x^2 + x}{3x^2 + 1}$ has the line $y = \frac{2}{3}$ as a horizontal asymptote.

(c) m > n. In this case, $\lim_{x \to \infty} f(x)$ and $\lim_{x \to \infty} f(x)$ do not exist. Hence the graph of f has no horizontal asymptote if the degree of the numerator of f exceeds the degree of the denominator of f. For example, the graph of the equation $y = \frac{2x^4}{x^2 - 1}$ has no horizontal asymptote. The case $m + 1 = n$ will be investigated later in Section 6. Notice that the graph of any rational function can have at most one horizontal asymptote. [There are, however, functions whose graphs possess more than one horizontal asymptote, e.g., $f(x) = \tan^{-1}x$; see Section 8.2 and, in particular, Figure 8.4 in the textbook.]

5. Vertical asymptotes shall be identified by the following definition. The line $x = a$ is called a vertical asymptote of the graph of a function f if there is a number a such that $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$. (See pages 43 and 45 in the textbook for a review of this type of limit.) Notice that necessarily f is defined in an open interval of the form (a, b), in the first case, or of the form (b, a), in the second case. Now it can be shown that $\lim_{x \to a} \frac{1}{(x - a)^p} = \pm\infty$ for all $p > 0$. Hence the line $x = a$ is a vertical asymptote of the graph of $f(x) = \frac{1}{(x - a)^p}$, $p > 0$. We shall derive here a more general result, of which this is a special case. Let $f(x) = \frac{g(x)}{h(x)}$, where g and h are continuous functions in some interval containing a, and such that $g(a) \neq 0$ and $h(a) = 0$. We claim that the line $x = a$ is a vertical asymptote of the graph of f. Indeed, since h is continuous at $x = a$, $\lim_{x \to a} h(x) = h(a) = 0$; that is,

to each number $\varepsilon > 0$ there is a number $\delta > 0$ such that $|h(x)| < \varepsilon$ for every x satisfying $0 < |x - a| < \delta$ [see the definition on page 40 of the textbook]. Hence, taking $\varepsilon = \frac{1}{N}$, for every $N > 0$ there exists a $\delta > 0$ such that $|h(x)| < \frac{1}{N}$ for every x satisfying $|x - a| < \delta$. Since $|h(x)| < \frac{1}{N}$ if and only if $\left|\frac{1}{h(x)}\right| > N$, it is easily seen that, according to the definition on page 45, $\underset{x \to a}{\text{limit}} \frac{1}{h(x)} = \pm\infty$. Hence, it follows* that $\underset{x \to a}{\text{limit}} \frac{g(x)}{h(x)} = \pm\infty$, and thus the line $x = a$ is a vertical asymptote of the graph of $f(x) = \frac{g(x)}{h(x)}$.

This leads to the following simple rule for finding the vertical asymptotes of the graph of a function f defined by $f(x) = \frac{g(x)}{h(x)}$. Set $h(x) = 0$ and solve for x; if $h(a) = 0$ and $g(a) \neq 0$, then the line $x = a$ is a vertical asymptote of the graph of f. [Note that the graph of a function may have arbitrarily many vertical asymptotes. For example, if N is any positive integer and if $a_1 \ldots a_N$ are distinct real numbers then the graph of the function f defined by

$$f(x) = \frac{1}{(x - a_1) \ldots (x - a_N)}$$

has the lines $x = a_1, \ldots, x = a_N$ as vertical asymptotes.]

6. There is another class of asymptotes which are important in the discussion of graphs; these are the so-called inclined asymptotes. The line $y = mx + b$ is called an inclined asymptote of the graph of a function f if

$$\underset{x \to \infty}{\text{limit}} [f(x) - (mx + b)] = 0$$

or

$$\underset{x \to -\infty}{\text{limit}} [f(x) - (mx + b)] = 0$$

or both. Intuitively, this means that the graph of f gets arbitrarily close to the line $y = mx + b$ when x is taken sufficiently large.

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*It can be proved that "if the function g is continuous at a and if $\underset{x \to a^+}{\text{limit}} f(x) = \pm\infty$ then $\underset{x \to a^+}{\lim} f(x) \cdot g(x)$

$$= \begin{cases} \underset{x \to a^+}{\text{limit}} f(x) \text{ if } g(a) > 0 \\ -\underset{x \to a^+}{\text{limit}} f(x) \text{ if } g(a) < 0. \end{cases}$$"

Consider a rational function f defined by $f(x) = \frac{g(x)}{h(x)}$, where g is of degree m and h is of degree m - 1. Suppose

$$f(x) = \frac{a_0x^m + a_1x^{m-1} + \dots + a_{m-1}x + a_m}{b_0x^{m-1} + b_1x^{m-2} + \dots + b_{m-2}x + b_{m-1}}$$

By long division we obtain

$$f(x) = \frac{a_0}{b_0}x + \frac{a_1b_0 - a_0b_1}{b_0^{\,2}} + \frac{g_1(x)}{h(x)},$$

where g_1 is a polynomial function of degree less than m - 1. Now

$$f(x) - \frac{a_0}{b_0}x + \frac{a_1b_0 - a_0b_1}{b_0^{\,2}} = \frac{g_1(x)}{h(x)}$$

and thus

$$\lim_{x\to\pm\infty} \left[f(x) - \left(\frac{a_0}{b_0}x + \frac{a_1b_0 - a_0b_1}{b_0^{\,2}}\right)\right] = \lim_{x\to\pm\infty} \frac{g_1(x)}{h(x)} = 0;$$

since the degree of g_1 is less than the degree of h. [See case 4 (a).] Hence the line

$$y = \frac{a_0}{b_0}x + \frac{a_1b_0 - a_0b_1}{b_0^{\,2}}$$

is an inclined asymptote of the graph of f.

We thus obtain the following result: if $f(x) = \frac{g(x)}{h(x)}$, where g and h are polynomial functions such that

degree of g = 1 + degree of h,

then the graph of f has an inclined asymptote. [Notice that in this case the graph of f does not have a horizontal asymptote.]

As an example, consider the graph of the equation

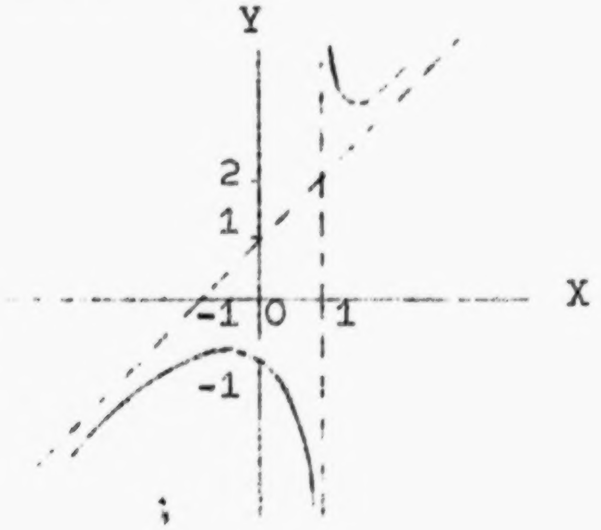
$$y = \frac{x^2 + 1}{x - 1}.$$

The degree of the numerator is 2 and the degree of the denominator is 1. Hence this graph has an inclined asymptote.

By long division,

$$\frac{x^2 + 1}{x - 1} = x + 1 + \frac{2}{x - 1},$$

so the inclined asymptote is the line $y = x + 1$. Note also that the line $x = 1$ is a vertical asymptote of the graph. The graph is shown below.



7. We shall close this lesson with an example which will illustrate some of the ideas discussed in the lesson and in the text.

Problem: Discuss and sketch the graph of the equation $y = \dfrac{x^2}{x^2 - 9}$.

Solution:

(1) Symmetry: With respect to the y-axis only

(2) Extent: Since $x = \pm\sqrt{\dfrac{y}{y - 1}}$ (check this) those values of y such that $\dfrac{y}{y - 1} \geq 0$ are only permissible i.e. $y > 1$ or $y \leq 0$.

(3) Intercept(s): when $x = 0$, $y = 0$ and when $y = 0$, $x = 0$ therefore the only intercept is at $(0, 0)$.

(4) Asymptotes: Since y is a rational function with degree of numerator equal to the degree of denominator we know (see 4(b)) $y = 1$ is a horizontal asymptote (or note that $\lim\limits_{x\to\infty} \dfrac{x^2}{x^2 - 9} = 1$.) To obtain the vertical asymptotes

we set $x^2 - 9 = 0$ (see 5.) and find $x = \pm 3$ as the vertical asymptotes (or note that $\lim_{x \to 3^+} \frac{x^2}{x^2 - 9} = \infty$ and $\lim_{x \to 3^+} \frac{x^2}{x^2 - 9} = \infty$.)

(5) Test For Relative Maximum and Minimum Points:

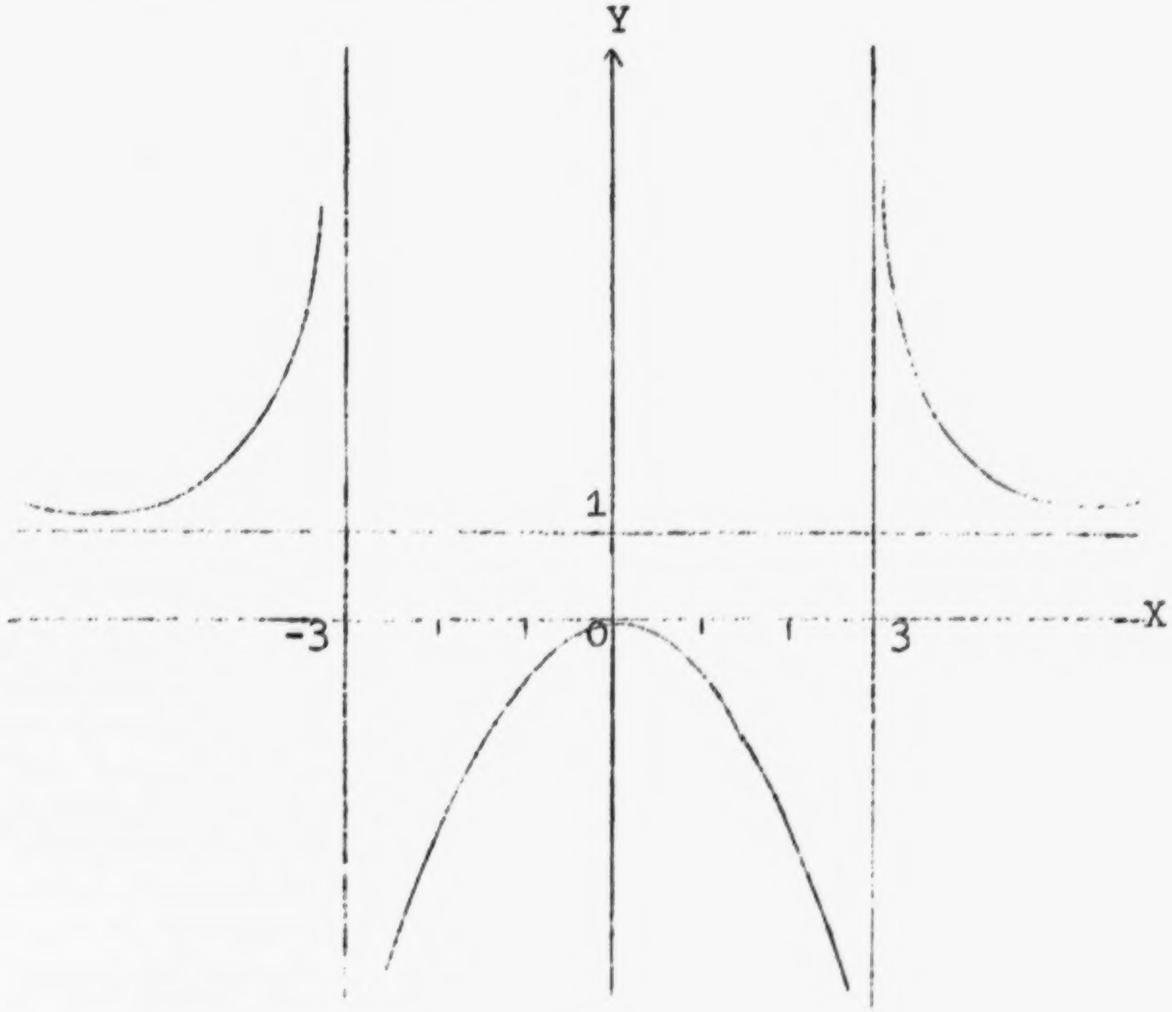
Since $f(x) = \frac{x^2}{x^2 - 9}$ we have

$f'(x) = \frac{18x}{(x^3 - 9)^2}$ (check this) and

$$f''(x) = \frac{-18(x^2 - 9)^2 + 72x^2(x^2 - 9)}{(x^2 - 9)^4}$$

$= \frac{(54x^2 + 162)}{(x^2 - 9)^3}$. Note that $f'(x) = 0$ when $x = 0$ and $f''(0) < 0$ therefore we have a relative maximum at $(0, 0)$. Note further that there are no points of inflection (Why?).

(6) Combining the results determined above we have the following sketch.



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Written Assignment

Solve Problems 1, 5, 6, 7, 10, page 158.

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Give the work for Lesson 16 to the supervisor who will complete the mailing procedure.

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Lesson 17

TANGENTS, NORMALS, NEWTON'S METHOD, DISTANCE BETWEEN TWO POINTS

Reference Material

Thomas, Chapter 4, Sections 2-5

Supplementary Explanation

1. Note in Section 2 that Equation (2), on page 158 of the text, is no longer valid if $f'(x_1)$ does not exist. Likewise Equation (3), on page 159, must carry the provision that $f'(x_1) \neq 0$. If $Q(c, d)$ is a point which does not lie on the graph of f and if there are non-vertical tangent lines to the graph of f passing through Q, their equations may be found in the following way. Consider an arbitrary point $P(x_1, f(x_1))$ on the graph of f. By (2), page 158 the tangent line to the graph of f at P has equation

   (i) $\quad y - f(x_1) = f'(x_1) \cdot (x - x_1).$

   Since $Q(c, d)$ is a point on this tangent line, its coordinates must satisfy (i). Hence

   (ii) $\quad d - f(x_1) = f'(x_1) \cdot (c - x_1).$

   If $x_1 = b$ is any solution of (ii) with $b \neq c$ and if $f(b)$ and $f'(b)$ exist, then, by (i) and (ii), $y - f(b) = f'(b)(x - b)$ is an equation of a tangent line to the graph of f passing through Q. If $b = c$ or if $f'(b)$ does not exist, the possibility of a vertical tangent line (with equation $x = b$) must be investigated.

   In a similar way we can find equations for the normal lines to the graph of f passing through a point Q which does not lie on the graph of f.

   As an illustration, consider the following problem (see also Example 2, page 160).

Find all tangent lines to the graph of the function $f(x) = \sqrt{x}$ which are on the point (0, 1).

Since $f'(x) = \frac{1}{2\sqrt{x}}$, an equation of the tangent line at any point $P(x_1, \sqrt{x_1})$, $x_1 \neq 0$, on the graph of f is

(iii) $\quad y - \sqrt{x_1} = \frac{1}{2\sqrt{x_1}}(x - x_1).$

[Note that f'(0) does not exist!] Since (0, 1) is a point on the tangent line, its coordinates must satisfy equation (iii); hence

$1 - \sqrt{x_1} = \frac{1}{2\sqrt{x_1}}(0 - x_1)$, or $\sqrt{x_1}\left(1 - \frac{\sqrt{x_1}}{2}\right) = 0$. Since $x_1 \neq 0$, the only solution of this equation is $x_1 = 4$.

Hence $y - 2 = \frac{1}{4}(x - 4)$ is an equation of the tangent line to the graph of $f(x) = \sqrt{x}$, passing through (0, 1).

Observe that the line $x = 0$ is another tangent to the graph of $f(x) = \sqrt{x}$, passing through (0, 1). In fact $x = 0$ is a vertical tangent. Give a rigorous definition of a vertical tangent to the graph of f at the point (a, f(a)).

2. We illustrate the use of the definition of the angle between two curves as given on page 162 (see also suggestion 4, Lesson 4). Example: Find the angle of intersection of the curves defined by the equations $x^2 + y^2 = 8$ and $x^2 = 2y$.

Solution: By solving the equations simultaneously we see that there are two points of intersection, namely (2, 2) and (-2, 2). By the symmetry of these curves with respect to the y-axis, it is evident that the angles of intersection of the curves is the same at each of these points. We will therefore determine the angle of intersection at the point (2, 2).

From the first equation we find that $2x + 2yy' = 0$, or $y' = -\frac{x}{y}$; therefore, the slope of the tangent to this curve at (2, 2) is $m_1 = -1$. From the second equation we obtain $2x = 2y'$, or $y' = x$, so that the slope of the tangent to this curve at (2, 2) is $m_2 = 2$. Therefore the angle θ between two tangents is given by

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1} = \frac{2 - (-1)}{1 + (2)(-1)} = -3.$$

Since $\tan\theta < 0$, θ must be a second quadrant angle. Therefore the angle between the two curves at the point (2, 2) is $180^\circ - \theta$. [Note that this angle, the supplement of θ, would have been obtained directly if we had taken $m_1 = 2$ and $m_2 = -1$.]

3. Consider Problem 11, page 164.

   Solution: Since $f(x) = x^2 + 2x - 3$ it follows that $f'(x) = 2x + 2$ so that $f'(1) = 4$. Thus the slope of the normal line at (1, 0) is $-\frac{1}{4}$. An equation of the normal at this point is given by $y = -\frac{1}{4}(x - 1)$. Solving this equation simultaneously with the original equation $y = x^2 + 2x - 3$ yields the second point of intersection, namely $\left(-\frac{13}{4}\ \ \frac{17}{16}\right)$.

4. One will immediately appreciate Newton's method for approximating roots (see Section 3) especially after a study of other methods (see for instance, Horner's method in a classical algebra text). Study carefully the examples given in this section. A more thorough discussion of this method should include a consideration of the sign of the second derivative in the interval under consideration. (See Analytic Geometry and Calculus by Adams and Whit[illegible], Oxford University Press 1961, pages 380-387). The subject will be covered lightly in this course but you may have occasion to use the method more extensively in the future.

5. The distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ was considered in Chapter 1 on page 6. In Sections 4, 5 and especially in later sections this formula will be used to determine many interesting and useful results. Section 4 provides one with another review of some important results from Trigonometry. Once again become very familiar with these results. We shall assume that you know them. In Section 5 the distance formula is used to solve so-called locus problems. We provide a further illustration.

   Problem 6, page 177

   Solution: Let P(x, y) be any point on the curve whose equation we desire. From the conditions of the problem and the distance formula we have

$$\sqrt{(x + 3)^2 + y^2} = \sqrt{(x - 3)^2 + y^2} + 4$$

$$\text{or } (x + 3)^2 + y^2 = (x - 3)^2 + y^2 + 16 + 8\sqrt{(x - 3)^2 + y^2}$$

or simplifying

$$3x - 4 = 2\sqrt{(x - 3)^2 + y^2}.$$

Squaring both sides we have

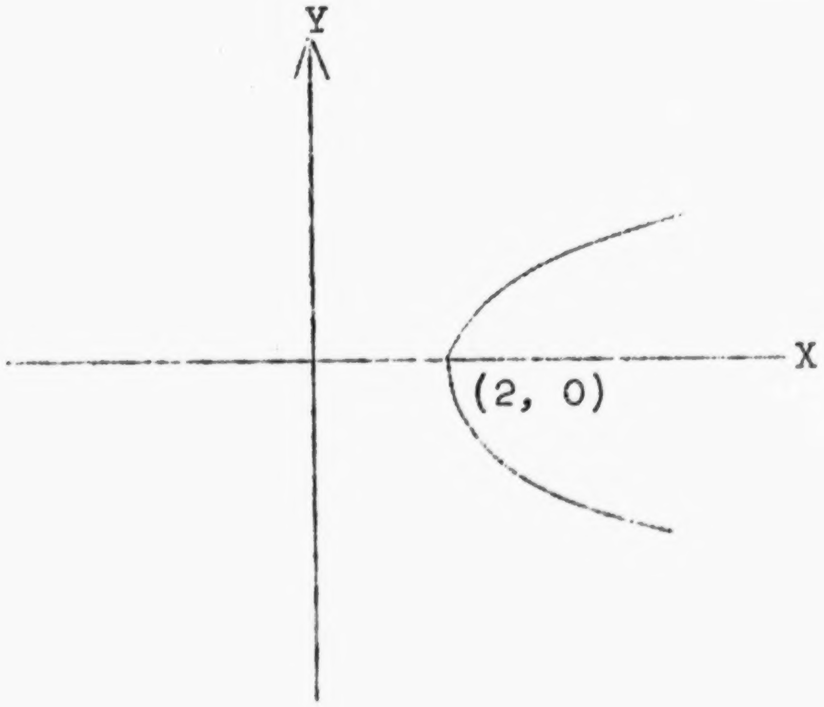
$$9x^2 - 24x + 16 = 4\ [(x - 3)^2 + y^2]$$

$$9x^2 - 24 + 16 = 4x^2 - 24x + 36 + 4y^2$$

and finally

$$5x^2 - 4y^2 = 20.$$

Now, not all points obtained from this equation belong to the original locus (Why?). In fact the graph of the locus is given by



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Written Assignment

Solve Problems 1, 3, 5, 7, 9, 13, 16, 21a, c, on pages 163-164. Solve Problems 1, 4, on page 169. Solve Problems 3, 5, 7, 9, page 175. Solve Problems 1, 3, 5, 8, 9, page 177.

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Give the work for Lesson 17 to the supervisor who will complete the mailing procedure.

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Lesson 18

THE CIRCLE AND THE PARABOLA

Reference Material

Thomas, Chapter 4, Sections 6 and 7

Supplementary Explanation

1. The so-called method of Completing the Square which was perhaps first encountered by you in elementary algebra (see your algebra text, in particular the proof of the quadratic formula) will be used extensively throughout the remainder of this chapter. If necessary, you should once again review this method.

2. Remark 1 on page 179 of Section 6 is rather important so that we should formulate it as a theorem.

Theorem: The graph of the equation

$$(1) \qquad Ax^2 + Ay^2 + Dx + Ey + F = 0 \qquad (A \neq 0)$$

is

(a) a circle if $D^2 + E^2 > 4AF$;
(b) a point if $D^2 + E^2 = 4AF$;
(c) imaginary if $D^2 + E^2 < 4AF$.

Conversely every circle is the graph of an equation of type (1).

Proof: Given equation (1), we complete the squares in x and y. [Since $A \neq 0$ we can divide by A.]

$$A\left(x^2 + \frac{D}{A}x + \frac{D^2}{4A^2}\right) + A\left(y^2 + \frac{E}{A}y + \frac{E^2}{4A^2}\right) = \frac{D^2}{4A} + \frac{E^2}{4A} - F$$

$$A\left(x + \frac{D}{2A}\right)^2 + A\left(y + \frac{E}{2A}\right)^2 = \frac{D^2 + E^2 - 4AF}{4A}$$

$$(2) \quad \left(x + \frac{D}{2A}\right)^2 + \left(y + \frac{E}{2A}\right)^2 = \frac{D^2 + E^2 - 4AF}{4A^2}$$

(a) If $D^2 + E^2 > 4AF$ then $D^2 + E^2 - 4AF > 0$, so the graph of (1) is a circle with center $\left(\frac{-D}{2A}, \frac{-E}{2A}\right)$ and radius

$$\frac{\sqrt{D^2 + E^2 - 4AF}}{2|A|}.$$

(b) If $D^2 + E^2 = 4AF$ then (2) becomes

$$\left(x + \frac{D}{2A}\right)^2 + \left(y + \frac{E}{2A}\right)^2 = 0$$

so $(x + \frac{D}{2A})^2 = 0$ and $(y + \frac{E}{2A})^2 = 0$, i.e., $x = \frac{-D}{2A}$, $y = \frac{-E}{2A}$.

Thus the graph of (1) consists of the single point $(\frac{-D}{2A}, \frac{-E}{2A})$.

(c) If $D^2 + E^2 < 4AF$ then the right-hand member of (2) is less than zero while the left-hand member, being the sum of two squares, is ≥ 0. Therefore no point lies on the graph of (1).

Conversely we have seen that a circle with center (h, k) and radius r has equation

$$(x - h)^2 + (y - k)^2 = r^2,$$

that is,

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0,$$

which is of type (1) with $A = 1$, $D = -2h$, $E = -2k$, $F = h^2 + k^2 - r^2$.

3. The equation

$$(x - 2)^2 + (y + 1)^2 = r^2,$$

where r is an unspecified real number (called a parameter), represents the family of all circles with center (2, -1). Similarly the equation

$$(x - h)^2 + (y - k)^2 = 16,$$

where h and k are parameters, represents the family of all circles with radius 4. Likewise the equation

$$(x - h)^2 + y^2 = r^2$$

represents the family of all circles with center on the x-axis (Why?), while the equation

$$x^2 + (y - k)^2 = r^2$$

represents the family of all circles with center on the y-axis. See if you can determine what family of circles is represented by the equation

$$(x - a)^2 + (y - a)^2 = r^2,$$

where a and r are parameters, and what family of circles is represented by the equation

$$x^2 + y^2 + ax + by = 0,$$

where a and b are parameters. Draw a few members of each of these families of circles, using a different set of axes for each family.

4. Further examples

(a) Problem 10, page 181

Solution: We are given

$2x^2 + 2y^2 + x + y = 0$. Upon completing the square we obtain

$$2(x^2 + \frac{x}{2} + \frac{1}{16}) + 2(y^2 + \frac{y}{2} + \frac{1}{16}) = 0 + \frac{1}{8} + \frac{1}{8}$$

or

$$2(x + \frac{1}{4})^2 + 2(y + \frac{1}{4})^2 = \frac{1}{4}$$

and finally

$$(x + \frac{1}{4})^2 + (y + \frac{1}{4})^2 = \frac{1}{8}.$$

Comparing with $(x - h)^2 + (y - k)^2 = r^2$ we find the center at $(-\frac{1}{4}, -\frac{1}{4})$ and radius $\sqrt{\frac{1}{8}}$.

(b) Problem 14, page 181

Solution: We are given $h = -1$, $k = 1$ so we need only find r, the radius. Since the circle is tangent to the line $x + 2y - 4 = 0$ and the tangent is perpendicular to the radius we may find the radius using equation (6) on page 34 of these notes. This formula will give the distance from the tangent line to the center $(-1, 1)$ which is precisely the radius. We obtain

$$r = \frac{|-1 + 2(1) - 4|}{\sqrt{1 + 4}} = \frac{|-3|}{\sqrt{5}} = \frac{3}{\sqrt{5}}.$$

Thus our desired equation is

$$(x + 1)^2 + (y - 1)^2 = \frac{9}{5}.$$

5. Learn the definition of the parabola and study very carefully the derivation of the various equations of the parabola given in Section 7. The important formulas are blocked off in the text. Notice the role that p plays with respect to the focus and directrix. If an equation of a parabola is not in one of the standard forms (see formulas 8a, b, c, d on page 186) then it may be changed to one of these forms again by the method of completing the square (see Example 3 on page 187). Why is it important for one to change an equation to a standard form?

6. Observe that a parabola has no asymptotes. For if the equation of a parabola is $x^2 = 4py$ or, equivalently, $y = \frac{x^2}{4p}$, then it is easily seen that for every number a,

$$\lim_{x \to a} \frac{x^2}{4p} = \frac{a^2}{4p},$$

so that there is no vertical asymptote. Further, since $\lim_{x \to \pm\infty} \frac{x^2}{4p}$ does not exist, there will be no horizontal asymptote. Finally no line $y = ax + b$ can be an inclined asymptote of a parabola since

$$\lim_{x \to \pm\infty} \left[\frac{x^2}{4p} - (ax + b)\right]$$

does not exist. [Similarly it may be verified that a parabola with equation $y^2 = 4px$ has no asymptote.]

We further observe that a parabola is bounded in a half-plane, but in no proper sector of a half-plane. Thus for example the parabola $y^2 = 4px$ with $p > 0$ is bounded in the half-plane consisting of all points (x, y) with $x \geq 0$ since this equation is satisfied only for non-negative values of x. On the other hand, to each non-negative value of x there corresponds at least one value of y and, conversely, to each value of y there corresponds a (unique) value of x.

7. We shall derive several more important properties of a parabola. Consider the parabola defined by the equation $y^2 = 4px$, $p > 0$, in connection with which see Figure 1.

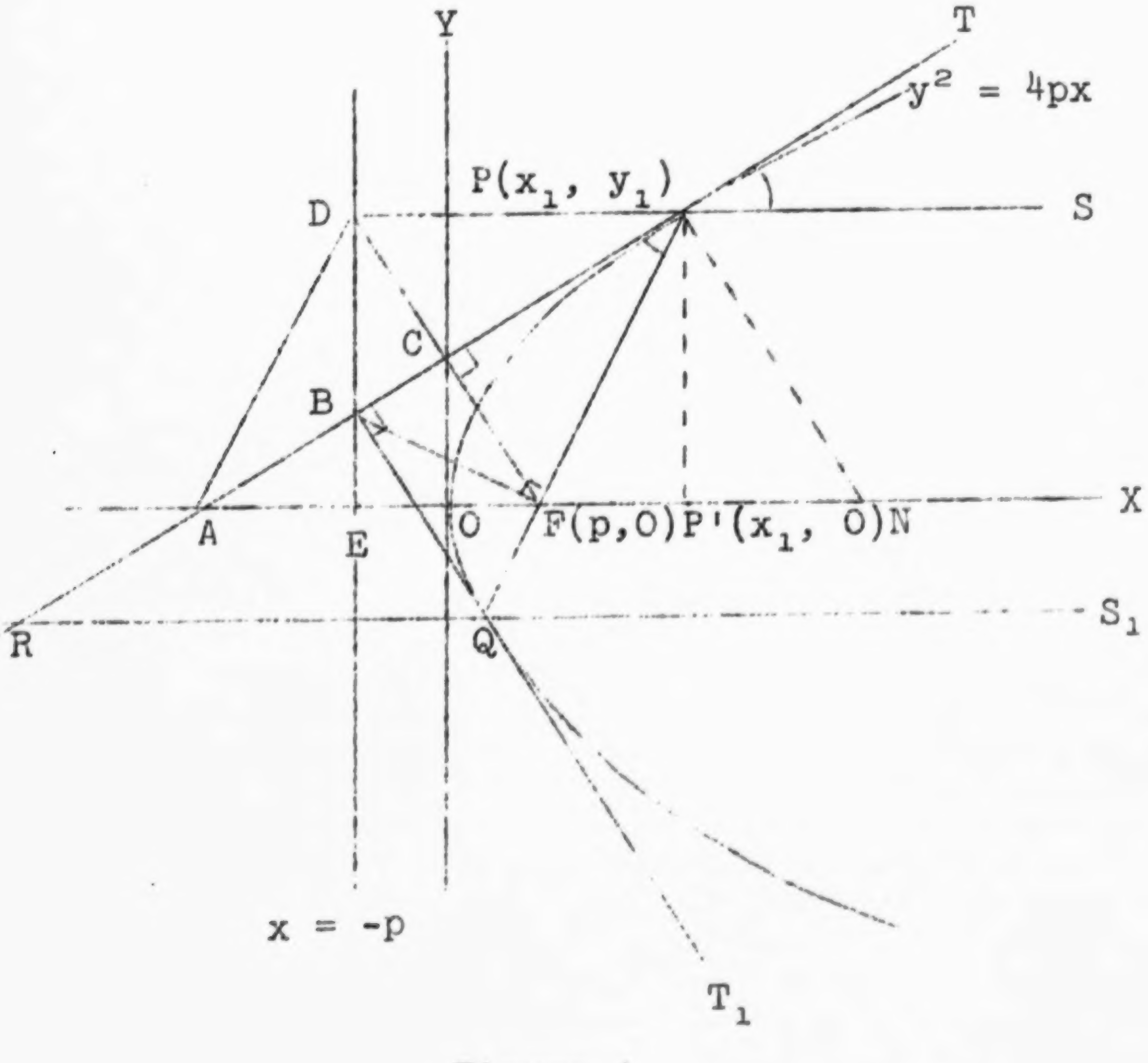

Figure 1

Let $F(p, 0)$ be the focus of this parabola, and let $P(x_1, y_1)$ be any point on the parabola distinct from the vertex. The segment FP is called a focal radius of the parabola. The chord drawn through F and P will meet the parabola in another point Q; the segment PQ is called a focal chord of the parabola. Let T and T_1 be the tangent lines to the parabola at P and Q, respectively; T intersects the tangent line at the vertex (i.e., the y-axis) at C, the directrix (i.e., the line $x = -p$) at B, and the axis of the parabola (i.e., the x-axis) at A. Let S and S_1 be the two lines parallel to the axis of the parabola and passing through P and Q, respectively. S intersects the directrix at D, and S_1 intersects the tangent line T at R. In terms of this notation we can make the following assertions.

(a) The quadrilateral AFPD is a rhombus (i.e., an equilateral parallelogram).

(b) FC is perpendicular to CP.

(c) BF is perpendicular to FP.

(d) The angle between T and S is equal to the angle between T and the focal radius FP. [See Problem 31, page 190.]

(e) T and T_1 intersect on the directrix and are perpendicular.

Proof: An equation of the tangent line T at $P(x_1, y_1)$ is given by $yy_1 = 2p(x + x_1)$ [verify this result; see Section 4-2]; thus T intersects the x-axis at $A(-x_1, 0)$. Let $P'(x_1, 0)$ be the projection of P onto the x-axis. Then O is the midpoint of the segment AP' and therefore, since the triangles AOC and AP'P are similar, C is the midpoint of the segment AP. Further, O is the midpoint of the segment EF, and thus C is also the midpoint of DF. Hence the diagonals AP and DF of the quadrilateral AFPD bisect each other, so AFPD is a parallelogram. Now the length of segment AF is $p + x_1$, and the length of segment FP is

$\sqrt{y_1^2 + (x_1 - p)^2} = \sqrt{4px_1 + (x_1 - p)^2} = x_1 + p$. Thus AF and FP have the same length so AFPD is a rhombus, proving (a).

Since the diagonals of a rhombus are perpendicular we obtain from (a) immediately that FD is perpendicular to AP and thus FC is perpendicular to CP, which proves (b).

The slope of the line FP is given by $m_{FP} = \frac{y_1}{x_1 - p}$; on the other hand, B has coordinates $(-p, \frac{2p}{y_1}(x_1 - p))$ (which are obtainable by solving simultaneously the equation of T: $yy_1 = 2p(x + x_1)$ and the equation of the directrix: $x = -p$). Hence the slope of the line FB is

$$m_{FB} = \frac{0 - \frac{2p}{y_1}(x_1 - p)}{p - (-p)} = -\frac{x_1 - p}{y_1}.$$

Since $m_{FP}m_{FB} = -1$, we see that FB is perpendicular to FP, proving (c).

By (a) segments AF and FP have the same length and therefore triangle AFP is isosceles. Hence the angles PAF and APF are equal. But the line S is parallel to the line AF and therefore the angle formed by S and T is equal to the angle PAF and hence equal to APF; this proves (d).

As in (c) we see that the tangent line T_1 at Q intersects the directrix at some point B_1 is perpendicular to FQ. Since also BF is perpendicular to PQ at F, $BF = B_1F$. Hence B, the point of intersection of BF with the directrix, coincides with B_1, the point of intersection of B_1F with the directrix. Now consider the triangle RQP. Since the angle between T and S_1 is equal to the angle between T and S, and since this latter angle is, by (d), equal to the angle between T and the line FP, the angles between T and S_1 and between T and the line QP are equal. Hence the triangle RQP is isosceles. By (d), the angle between T_1 and the line QF is equal to the angle between T_1 and S_1; hence the line BQ bisects the angle RQP. Since triangle RQP is isosceles, BQ is perpendicular to RP. This proves (e).

8. Final Example

Problem 27, page 190

Solution: Obviously, from the given figure the appropriate standard form is given by

$(x - h)^2 = -4p(y - k)$. The vertex of the given parabola is at $(\frac{b}{2}, h)$ and thus an equation becomes

$(x - \frac{b}{2})^2 = -4p(y - h)$. There remains to find p (or $-4p$). Since the parabola passes through (0, 0) this point must satisfy its equation. Hence we have

$(0 - \frac{b}{2})^2 = -4p\ (0, h)$ from which we find

$-4p = -\frac{b^2}{4h}$. Thus an equation of the desired parabola is given by

$$(x - \frac{b}{2})^2 = -\frac{b^2}{4h}(y - h), \text{ which reduces to}$$

$$b^2y = 4h\, x(b - x).$$

* * * * *

Written Assignment

Solve Problems 3, 6, 7, 9, 12, 13, 15, 17, 20, 21, pages 181-182 in the textbook. Solve Problems 3, 6, 7, 9, 13, 17, 21, 23, 26, 28a, 30, pages 189-190.

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Give the prepared work for Lesson 18 to the supervisor who will complete the mailing procedure.

* * * * *

Lesson 19

THE ELLIPSE AND HYPERBOLA

Reference Material

Thomas, Chapter 4, Sections 8 and 9

Supplementary Explanation

1. The definitions of the ellipse and hyperbola should be learned.

2. The essential results associated with the ellipse (Section 8) may be summarized as follows:

   a. The Standard Forms:

   $$\text{(i)}\quad \frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

   $$\text{(ii)}\quad \frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$$

   (i) represents an ellipse with center (h, k) and major axis parallel to or lying along the x-axis. (ii) represents an ellipse with center (h, k) and major axis parallel to or lying along the y-axis.

   b. Keep in mind the following results:

   $$a^2 = b^2 + c^2.$$

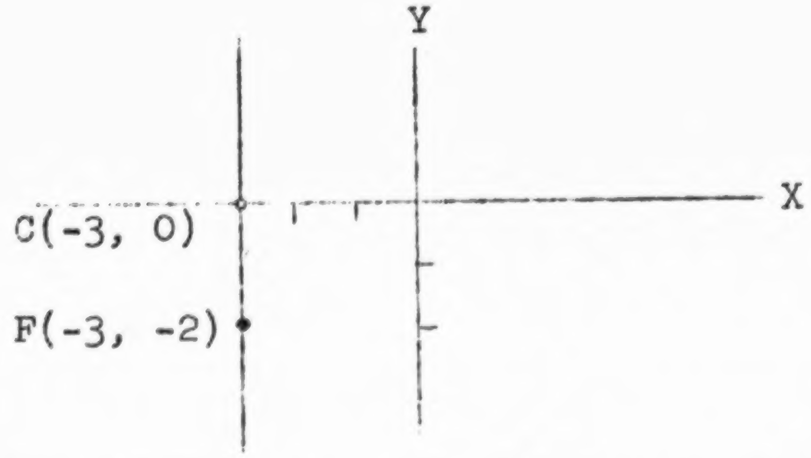
   c. If you are given an equation not already in standard form one can algebraically convert it to a standard form by completing the square, (see Example 1, page 193). If the equation is in standard form then it can be easily analyzed. Note that its position is always determined by the value of a, and a is always greater than b. (See Figure 4-39, page 195.) Note that when a = b we have a circle.

d. The bound on the eccentricity for the various "conics" are given at the bottom of page 196. The eccentricity of the ellipse actually is bounded thus, $0 < e = \frac{c}{a} < 1$, since $0 < c < a$. The eccentricity of the circle is 0. (See Figure 4-40, page 196.)

3. Consider the following problems.

a. Problem 4, page 198

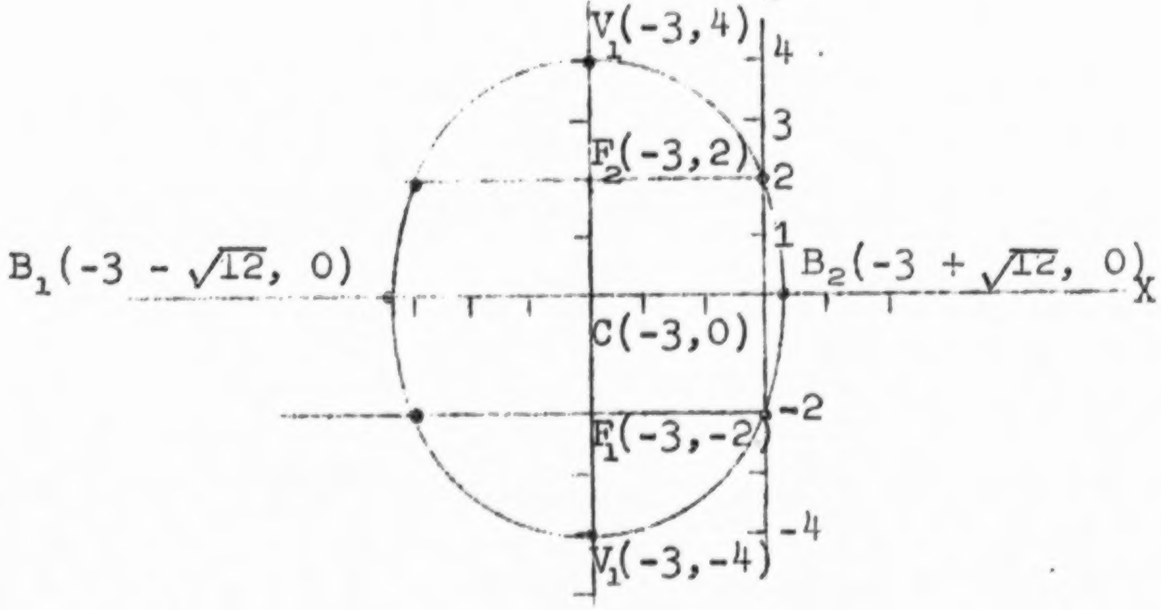
Solution: We first sketch the given points.



Now from the figure it is clear that the major axis will be parallel to the y-axis, c = 2 and we are given a = 4. Since $a^2 = b^2 + c^2$ we have $b^2 = 12$. An equation of the ellipse is given by

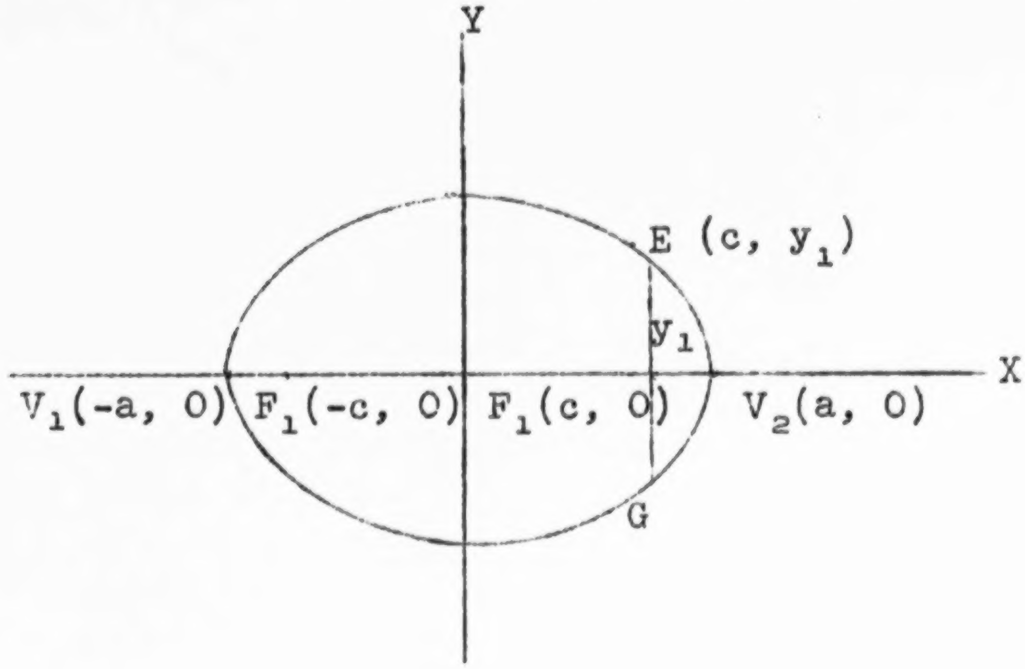
$$\frac{(y - 0)^2}{16} + \frac{(x + 3)^2}{12} = 1, \quad e = \frac{c}{a} = \frac{2}{4} = \frac{1}{2}.$$

Our sketch:



b. Problem 12, page 198

Solution: Our ellipse is in the following position and we wish to determine the length of the chord GE.



Now by symmetry $GE = 2F_1E$. If we let $F_1E = y_1$ then $GE = 2y_1$. We shall determine y_1 in the following manner. The coordinates of E are obviously given by (c, y_1). Since this point lies on the ellipse we have

$$b^2c^2 + a^2y_1^2 = a^2b^2 \text{ or}$$

$$a^2y_1^2 = a^2b^2 - b^2c^2 = b^2(a^2 - c^2) = b^2 \cdot b^2$$

since $a^2 = b^2 + c^2$.

Thus

$$y_1^2 = \frac{b^4}{a^2} \text{ and}$$

$$y_1 = \pm\frac{b^2}{a}.$$

We take $y_1 = +\frac{b^2}{a}$ since by assumption (see diagram) $y > 0$.

Now $GE = 2y_1 = \frac{2b^2}{a}$. This is called the length of the latus rectum of the ellipse. One immediate use of the latus rectum is found in sketching ellipses, four points of an ellipse are immediately determined by this chord. For Problem 4, page 198 we have for the length of the latus rectum $\frac{2b^2}{a} = \frac{2.12}{4} = 6$. The latera recta for this ellipse are plotted in the diagram.

4. The coordinates of a point $P(x, y)$ which is on the ellipse with foci $F(c, 0)$ and $F'(-c, 0)$, with $|FP| + |F'P| = 2a$, must satisfy the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b^2 = a^2 - c^2$. Hence if $P(x, y)$ is on this ellipse, then the coordinates of P must satisfy

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2) \text{ and } x^2 = \frac{a^2}{b^2}(b^2 - y^2).$$

It follows immediately that $|x| \leq a$ and $|y| \leq b$. Thus an ellipse is bounded in a rectangular region (with sides of lengths $2a$ and $2b$, respectively).

5. We shall derive some further properties of the ellipse. Consider an ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose foci are $F(c, 0)$ and $F'(-c, 0)$. Let C_2 and C_3 designate the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$, respectively. Let a ray, originating at the origin and making an angle θ with the x-axis, intersect these two circles at P_2 and P_3, respectively. Further, let the tangent to the circle C_2 at the point P_2 intersect the x-axis at T_2, and let the tangent to the circle C_3 at the point P_3 intersect the y-axis at T_3. Finally, let P_1 be the point of intersection of the line through P_2 and parallel to the y-axis with the line through P_3 and parallel to the x-axis.

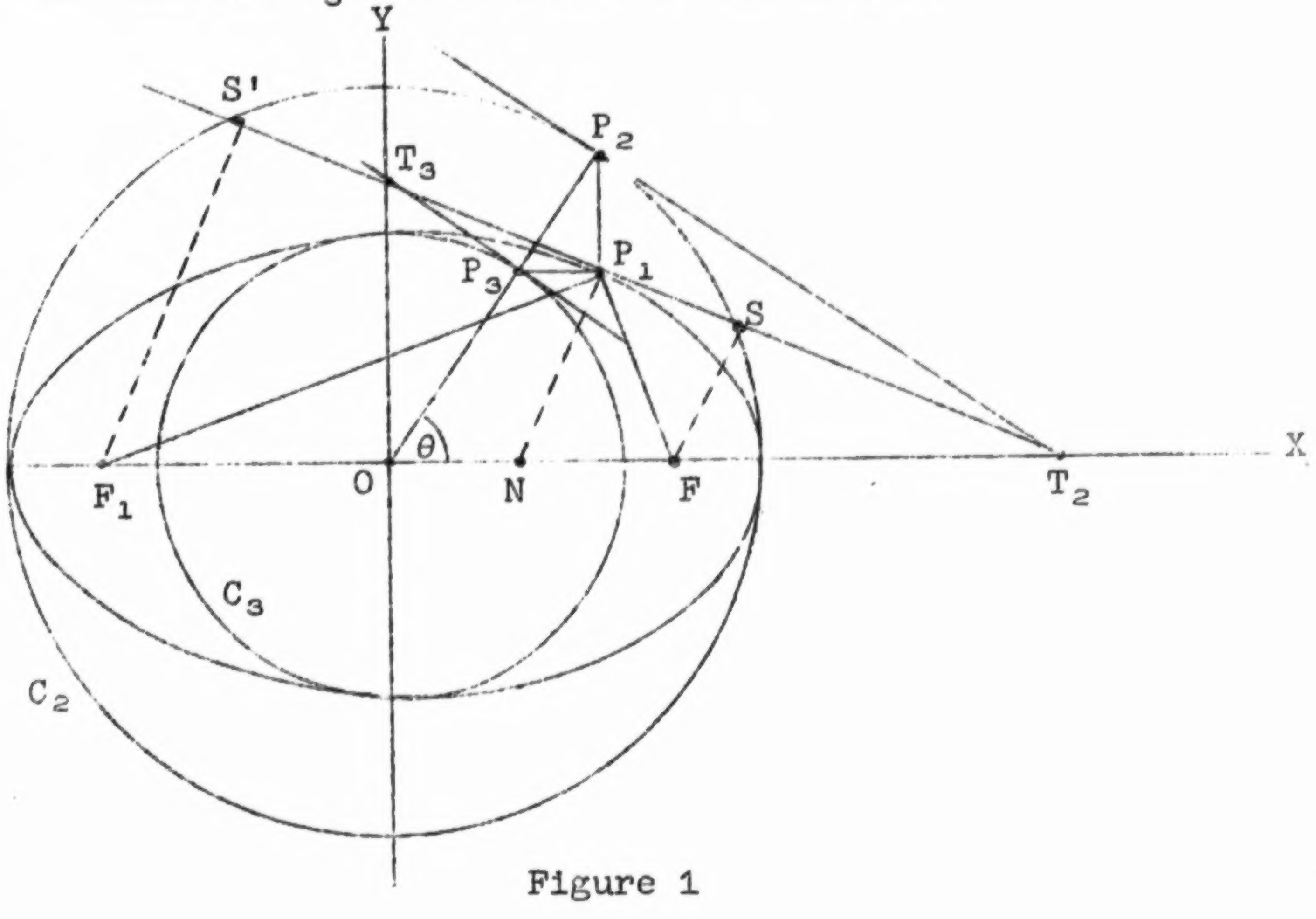


Figure 1

We will prove the following statements.

(a) P_1 is a point of the ellipse.

(b) The tangent line to the ellipse at P_1 passes through T_2 and T_3.

(c) The focal radii $F'P_1$ and FP_1 make equal angles with the tangent to the ellipse at P_1.

Proof: The abscissa of P_2 is a cos θ and the ordinate of P_3 is b sin θ. Hence P_1 has abscissa $x_1 = a \cos \theta$ and ordinate $y_1 = b \sin \theta$.

Now

$$\frac{x_1{}^2}{a^2} + \frac{y_1{}^2}{b^2} = \cos^2 \theta + \sin^2 \theta = 1,$$

and thus P_1 is a point of the ellipse, proving (a). Observe that, conversely, every point of the ellipse may be obtained in the manner described for the construction of P_1. This then gives a useful mechanical construction for the points of an ellipse.

An equation of the tangent to the ellipse at the point $P_1(x_1, y_1)$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$. [See Exercise 15, page 198.] Thus this tangent line intersects the x-axis and the y-axis at the points $(\frac{a^2}{x_1}, 0)$ and $(0, \frac{b^2}{y_1})$, respectively. [We assume here that neither x_1 nor y_1 is 0; this special case is easily dealt with separately.] The tangent to circle C_2 at the point $P_2(x_2, y_2)$ has equation $xx_2 + yy_2 = a_2{}^2$, and hence the coordinates of T_2 are $(\frac{a^2}{x_2}, 0)$. Similarly the tangent to circle C_3 at $P_3(x_3, y_3)$ has equation $xx_3 + yy_3 = b^2$, so T_3 has coordinates $(0, \frac{b^2}{y_3})$. Since $x_1 = x_2$ and $y_1 = y_3$, we see that the tangent to the ellipse at $P_1(x_1, y_1)$ intersects the x-axis and the y-axis at T_2 and T_3, respectively. This proves (b).

Let N be the intersection of the normal to the ellipse at P_1 with the x-axis. An equation of this normal is

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1).$$

[Notice that the tangent to the ellipse at P_1 has slope $\frac{-b^2x_1}{a^2y_1}$, so the normal to the ellipse at P_1 has slope $\frac{a^2y_1}{b^2x_1}$.]

Setting $y = 0$ we obtain for the abscissa of N:

$$x = \frac{a^2 - b^2}{a^2} x_1 = \frac{c^2}{a^2} x_1.$$

Hence

$$\frac{|NF|}{|NF'|} = \frac{c - \frac{c^2}{a^2} x_1}{\frac{c^2}{a^2} x_1 + c} = \frac{a^2 - cx_1}{cx_1 + a^2}.$$

The lengths of the focal radii FP_1 and $F'P_1$ are given by

$$|FP_1| = \frac{1}{a}(a^2 - cx_1) \text{ and } |F'P_1| = \frac{1}{a}(cx_1 + a^2).$$

[See (5), page 192.] Hence we have

$$\frac{|NF|}{|NF'|} = \frac{|FP_1|}{|F'P_1|}.$$

Thus for triangle $F'P_1F$ the normal line divides the side $F'F$ proportionally to the sides $F'P_1$ and FP_1. This means that the normal line bisects the angle $F'P_1F$. Thus the angles $F'P_1N$ and FP_1N are equal. From this (c) follows immediately.

We here mention an additional property: If S and S' are the projections of F and F', respectively, on the tangent to the ellipse at P_1, then S and S' lie on the circle C_2; moreover $|F'S'| \cdot |FS| = b^2$. Verify this property.

6. Study very carefully the text and illustrative examples associated with the hyperbola as given in Section 9. Note the specific difference between the results associated with the ellipse and hyperbola. For the hyperbola we have the standard forms

(i) $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$; center at (h, k) and transverse axis parallel to or lying along the x-axis;

(ii) $\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$; center at (h, k) and transverse axis parallel to or lying along the y-axis.

(iii) For the hyperbola:

$$c^2 = a^2 + b^2$$

$$e = \frac{c}{a} > 1 \text{ since } c > a.$$

Length of latus rectum = $\frac{2b^2}{a}$ (proof similar to that given for Problem 12, page 198, see Section 3b). The expressions for eccentricity and length of the latus rectum are precisely the same for both the ellipse and hyperbola, but of course their numerical values will not be the same because of the different relationship between a, b and c for these conics. Incidently, what is the length of the latus rectum for the parabola $y^2 = 4px$?

(iv) The equations of the asymptotes for the hyperbola are obtained from either

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 0$$

or

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 0.$$

These asymptotes are the diagonals of the so-called fundamental rectangle determined by the lines $x = \pm a$, $y = \pm b$ or $y = \pm a$, $x = \pm b$ (depending upon the position of the hyperbola).

(v) For a discussion of directricies see page 205.

7. Consider the following problems.

   a. Problem 7, page 206

   Solution: We are given

   $4y^2 = x^2 - 4x$ or

   $x^2 - 4x - 4y^2 = 0.$

   Completing the square, we have

   $x^2 - 4x + 4 - 4y^2 = 4$

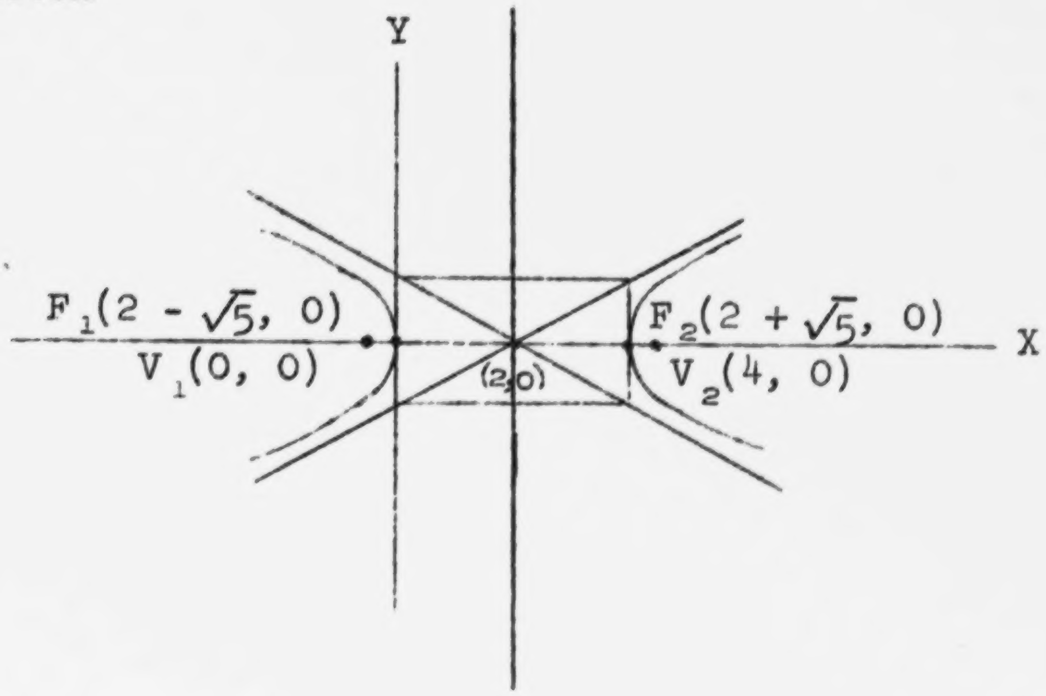
   $(x - 2)^2 - 4y^2 = 4.$

   Putting this in standard form:

   $$\frac{(x - 2)^2}{4} - \frac{y^2}{1} = 1.$$

Thus the transverse axis parallel to the x-axis and $a = 2$, $b = 1$, $c^2 = a^2 + b^2 = 4 + 1 = 5$, $c = \pm\sqrt{5}$, center at $(2, 0)$; vertices at $(4, 0)$ and $(0, 0)$; foci at $(2 - \sqrt{5}, 0)$ and $(2 + \sqrt{5}, 0)$; equations of the asymptotes are given by $\frac{(x - 2)^2}{4} - \frac{y^2}{1} = 0$.

Sketch



b. Problem 10, page 206

Solution: We shall leave the complete solution of this problem to the reader. If necessary you may use the following hints.

(i) For the equation to represent an ellipse we must simultaneously have $9 - C > 0$ and $5 - C > 0$.

(ii) For the equation to represent a hyperbola either $9 - C > 0$ and $5 - C < 0$ or $9 - C < 0$ and $5 - C > 0$.

(iii) For the ellipse $a^2 = b^2 + c^2$; for the hyperbola $c^2 = a^2 + b^2$.

8. Observe that a hyperbola is bounded by the two sectors formed by its asymptotes. Thus if $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is an equation of the hyperbola, then the hyperbola is bounded by its asymptotes $y = \pm\frac{b}{x} x$ in such a way that for any point $P(x_1, y_1)$ on the hyperbola, $|y_1| < \frac{b}{a}|x_1|$; that is, for $x_1 > 0$ all points $P(x_1, y_1)$ of the hyperbola lie below the asymptote $y = \frac{b}{a} x$ and above the asymptote $y = -\frac{b}{a} x$, while

for $x_1 < 0$ all points $P(x_1, y_1)$ of the hyperbola lie below the asymptote $y = -\frac{b}{a}x$ and above the asymptote $y = \frac{b}{a}x$.

9. Consider the hyperbola defined by the definition

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

with foci $F(x, 0)$ and $F'(-c, 0)$. If $P(x_1, y_1)$ is any point on the hyperbola, then the tangent to the hyperbola at P bisects the angle F'PF between the focal radii F'P and FP.

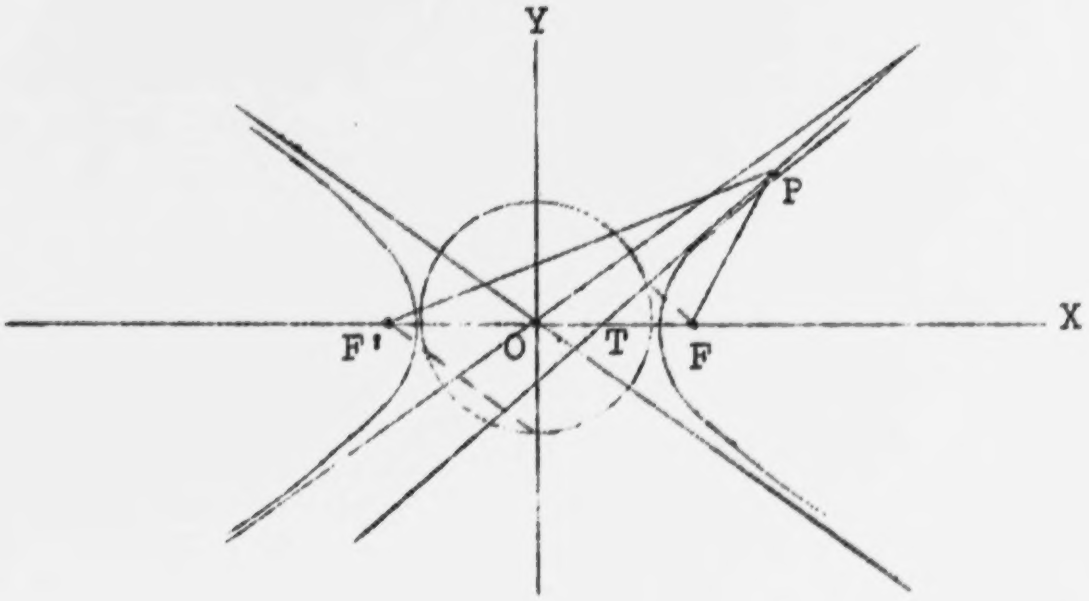


Figure 1

Let T be the intersection of the tangent line to the hyperbola at P with the x-axis. Since this tangent line has equation

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1,$$

T has coordinates $(\frac{a^2}{x_1}, 0)$. Moreover, since $|x_1| > a$, T clearly lies between F and F'. Now

$$\frac{|F'T|}{|FT|} = \frac{\left|\frac{a^2}{x_1} + c\right|}{\left|\frac{a^2}{x_1} - c\right|} = \frac{|a^2 + cx_1|}{|a^2 - cx_1|} = \frac{|F'P|}{|FP|}.$$ (See 4a, b page 200.)

Hence in the triangle F'PF the point T divides the side F'F proportionally to the sides F'P and FP. Hence TP bisects angle F'PF.

* * * * *

Written Assignment

Solve Problems 5, 6, 7, 8c, 9, 10, 13, 14, 15, page 198 in the textbook. Solve Problems 1c, d, 3, 8, 11, 12, page 206 in the textbook.

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Give the work for Lesson 19 to the supervisor who will complete the mailing procedure.

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Lesson 20

TRANSLATION, ROTATION, SECOND DEGREE CURVES, INVARIANTS

Reference Material

Thomas, Chapter 4, Sections 10, 11, 12

Supplementary Explanation

1. On page 185 of the text the notion of translation of axes was first discussed. We make a few additional remarks concerning this concept.

2. Consider the general second degree equation in x and y:

   (1) $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$,

   where A, B, and C are not all zero. Under the translation of axes defined by

   (2) $x = x' + h,\ y = y' + k$,

   the equation (1) is transformed into

   $$A(x' + h)^2 + B(x' + h)(y' + k) + C(y' + k)^2 + D(x' + h) + E(y' + k) + F = 0$$

   or

   $$(3)\ Ax'^2 + Bx'y' + Cy'^2 + (2Ah + Bk + D)x' + (Bh + 2Ck + E)y' + (Ah^2 + Bhk + Ck^2 + Dh + Ek + F) = 0.$$

   Upon comparison of equations (1) and (3) we see that the coefficients of the second degree terms of (1) are unchanged under the translation (2). It can be shown that for any polynomial equation in x and y the coefficients of the terms of highest degree are invariant (i.e., left fixed) under any translation.

   We note that the equations of the conic sections as presented in Sections 4.7-4.9 are of second degree, and hence under any translation or axes these equations will remain of second degree.

3. Under certain conditions the linear terms of equation (1) may be removed by a suitable translation of axes. From (3) it follows that h and k must be chosen in (2) so as to satisfy the pair of equations

$$(4)\ \begin{cases} 2Ah + Bk + D = 0, \\ Bh + 2Ck + E = 0. \end{cases}$$

This system of simultaneous linear equations in the unknowns h and k will have a unique solution if and only if

$$\begin{vmatrix} 2A & B \\ B & 2C \end{vmatrix} \neq 0,$$

in which case the solution of (4) is obtained from the determinants

$$h = \frac{\begin{vmatrix} -D & B \\ -E & 2C \end{vmatrix}}{\begin{vmatrix} 2A & B \\ B & 2C \end{vmatrix}}, \quad k = \frac{\begin{vmatrix} 2A & -D \\ B & -E \end{vmatrix}}{\begin{vmatrix} 2A & B \\ B & 2C \end{vmatrix}}.$$

Using these values for h and k in (2) yields a translation which will eliminate the linear terms in (1).

If in (1) $B = 0$, then one may eliminate the linear terms by completing the squares in x and y, as illustrated in Example 2 on page 203.

4. Consider the following example. Remove the linear terms of the following equation by a suitable translation of axes. $xy + 4x - y = 5$

Solution: Letting $x = x' + h$, $y = y' + k$ our equation becomes

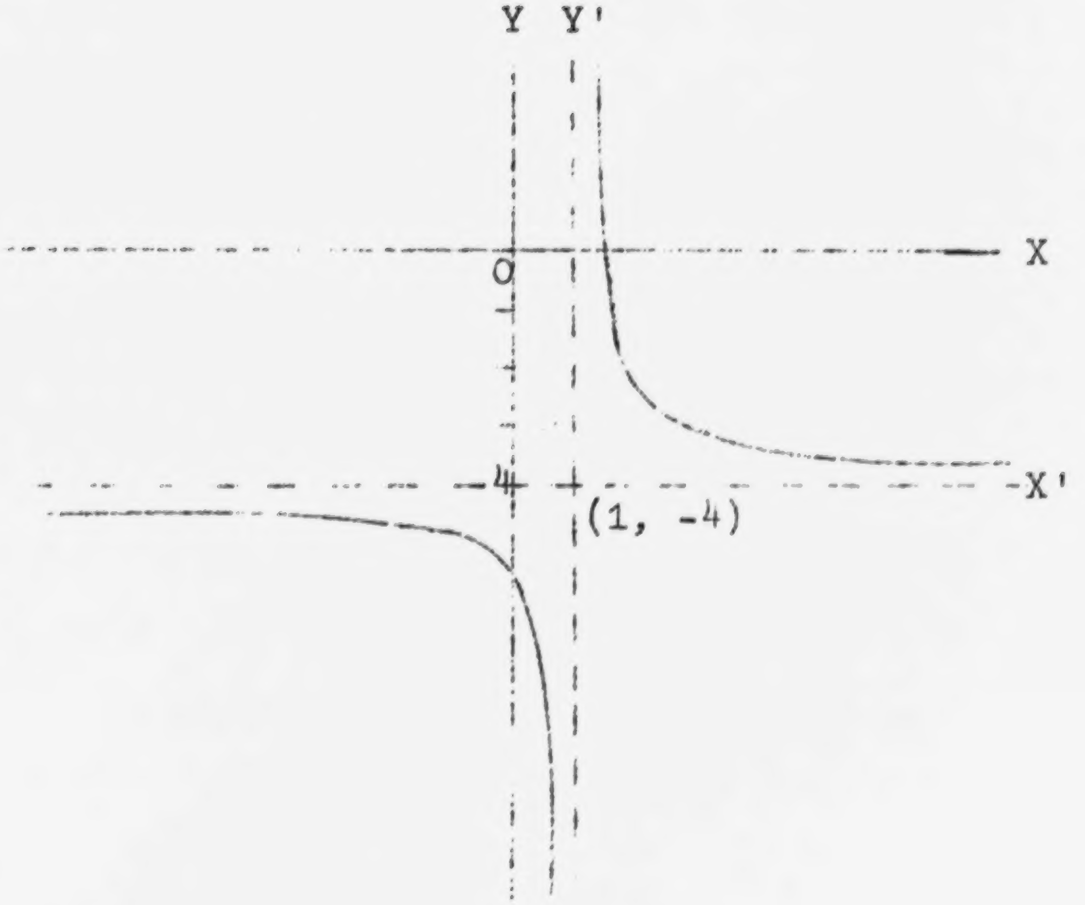
$(x' + h)(y' + k) + 4(x' + h) - (y' + k) = 5$ or
$x'y' + kx' + hy' + hk + 4x' + 4h - y' - k = 5$ and finally
$x'y' + (k + 4)\, x' + (h - 1)\, y' + hk + 4h - k = 5.$

We may rid of the x' and y' terms if we let $k + 4 = 0$ and $h - 1 = 0$ or if we take $k = -4$ and $h = 1$. Thus our new equation reads

$x'y' + 0 + 0 - 4 + 4 + 4 = 5$ or
$x'y' = 1.$

If this equation is now graphed with respect to the new origin at (1, -4) the resulting curve will be the same as that obtained by plotting the original equation with (0, 0) as origin.

Sketch:



Note that H and k could have been determined directly from the equations in (4) found in 3 above.

5. Regarding rotation of axes consider again the general second degree equation in x and y:

(1) $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

Using the rotation of axes defined by (see (4) page 208)

$$\begin{cases} x = x' \cos\alpha - y' \sin\alpha \\ y = x' \sin\alpha + y' \cos\alpha \end{cases}$$

the equation (1) is transformed into

$$A(x' \cos\alpha - y' \sin\alpha)^2$$

$$+ B(x' \cos\alpha - y' \sin\alpha)(x' \sin\alpha + y' \cos\alpha)$$

$$+ C(x' \sin\alpha + y' \cos\alpha)^2 + D(x' \cos\alpha - y' \sin\alpha)$$

$$+ E(x' \sin\alpha + y' \cos\alpha) + F = 0,$$

or

(2) $A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$,

where

$$(3)\begin{cases} A' = A\cos^2\alpha + B\sin\alpha\cos\alpha + C\sin^2\alpha, \\ B' = 2(C - A)\sin\alpha\cos\alpha + B(\cos^2 + \theta - \sin^2\theta), \\ C' = A\sin^2\alpha - B\sin\alpha\cos\alpha + C\cos^2\alpha, \\ D' = D\cos\alpha + E\sin\alpha \\ E' = E\cos\alpha - D\sin\alpha, \\ F' = F. \end{cases}$$

It is easily seen from (3) that

$$(4)\ A' + C' = A + C$$

and that

$$B'^2 - 4A'C' = B^2 - 4AC.$$

Hence in equation (1) the quantities F, $A + C$, and $B^2 - 4AC$ are invariant under any rotation of axes.

It follows that under any rotation of axes the degree of equation (1) is unchanged. For suppose that equation (2) is of degree less than 2 (clearly the degree of (1) cannot be increased by a rotation of axes). Then $A' = B' = C' = 0$ and hence $A + C = A' + C' = 0$ or $C = -A$. Also $B^2 - 4AC = B^2 + 4A^2 = 0$ and therefore $B = 2A = 0$ (since a sum of squares of real numbers is zero if and only if each of the numbers is zero). Therefore $A = B = C = 0$, which is contrary to our assumption that (1) is of the second degree.

Thus in particular the equations of the conic sections as given in 4.7-4.9 will be of second degree after any rotation of axes.

In order to eliminate the xy term of equation (1) by a rotation of axes we must choose the angle of rotation, α, such that in (3) $B' = 0$. As indicated in the discussion on page 209, α must be chosen so that

$$(5)\ \cot 2\alpha = \frac{A - C}{B}$$

(assuming that $B \neq 0$).

Now since $\cot 2\alpha = \dfrac{\cot^2\alpha - 1}{2\cot\alpha}$ (5) may be written as

$$\frac{\cot^2\alpha - 1}{2\cot\alpha} = \frac{A - C}{B} \text{ or}$$

$\cot^2\alpha - 1 = \dfrac{2(A - C)}{B}\cot\alpha$ which is the same as

$\cot^2\alpha - \frac{2(A - C)}{B}\cot\alpha - 1 = 0$. Using the quadratic formula we have

$$\cot\alpha = \frac{\frac{2(A - C)}{B} \pm \sqrt{4\left(\frac{A - C}{B}\right)^2 + 4}}{2}$$ and finally

$$(6)\quad \cot\alpha = \frac{A - C}{B} \pm \sqrt{\left(\frac{A - C}{B}\right)^2 + 1}$$

The ± in this last formula indicates that in general we will have two solutions, one of which will be an acute angle since $\sqrt{(A - C)^2 + B^2} > |A - C|$, and so one value of the cot α will be positive and the other negative.

We next prove

$$(7)\quad A' - C' = B\sqrt{\left(\frac{A - C}{B}\right)^2 + 1}$$

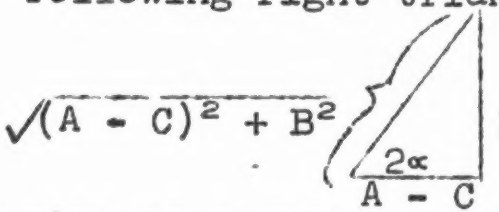
Proof:

From (3) we have

$A' - C' = (A - C)\cos 2\alpha + B\sin 2\alpha$ or

$$\begin{aligned} A' - C' &= B\cot 2\alpha \cdot \cos 2\alpha + B\sin 2\alpha \text{ (here we used (5))} \\ &= B\frac{\cos^2 2\alpha}{\sin 2\alpha} + B\sin 2\alpha \\ &= B\frac{1}{\sin 2\alpha} \\ &= B\frac{1}{\frac{B}{\sqrt{(A - C)^2 + B^2}}} \\ &= B\frac{\sqrt{(A - C)^2 + B^2}}{B} \\ &= B\sqrt{\left(\frac{A - C}{B}\right)^2 + 1} \end{aligned}$$

[This follows from the fact that from (5) $\cot 2\alpha = \frac{A - C}{B}$ and thus we may obtain sin 2α from the following right triangle



].

Now equations (4) and (7) will give us a rapid method for obtaining A' and C' in any rotation problem.

Example: Perform a rotation of axes that will eliminate the xy term of $7x^2 + 12xy - 2y^2 = 10$.

Solution: Assuming we rotate through the angle determined by (6) we first note that $B' = 0$ and $F' = -10$. To determine A' and B' we use (4) and (7). From the given equation we know $A = 7$, $B = 12$ and $C = -2$. Thus from (4) and (7) we have

$A' + C' = 7 - 2 = 5$

$$A' - C' = 12\sqrt{\left(\frac{7-(-2)}{12}\right)^2 + 1} = 12\sqrt{\frac{81+144}{144}} = 15.$$

$$\left.\begin{aligned} \text{Hence } A' + C' &= 5 \\ A' - C' &= 15 \end{aligned}\right\}$$ Upon first adding then subtracting these

two equations we obtain $A' = 10$ and $C' = -5$.

Our new equation reads

$10x'^2 - 5y'^2 - 10 = 0$ or

$2x'^2 - y'^2 - 2 = 0$ or

$x'^2 - \frac{y'^2}{2} = 1.$

One can easily plot this equation with respect to the new x'y' axes obtained by rotation of the original axes through the angle α determined from (6).

6. Consider once again the general second degree equation in x and y:

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$

where A, B, C, D, E, F are real numbers with A, B, C not all zero. We have seen that under any translation or rotation of axes the degree of this equation is unchanged. Since the conic sections as presented in 4.7-4.9 have equations of second degree, it follows that any conic section is defined by a (polynomial) equation of second degree, no matter how the coordinate axes are chosen relative to the conic section. We shall be concerned now with the converse problem, that of determining the graph of any second degree polynomial equation in x and y. We shall prove that, with some rather trivial exceptions, the graph of any such equation is one of the conic sections.

First we observe that not all polynomial equations of the second degree have graphs; for example, the equation $x^2 + y^2 + 2 = 0$ has no graph since $x^2 + y^2 + 2 > 0$ for all real numbers x and y. Second, there are equations of the second degree whose graphing consist of a single point, as is the case for the equation $x^2 + y^2 = 0$, the graph of which consists of the point (0, 0).

Third, it can happen that an equation of the second degree is factorable, i.e., can be written in the form

(1) $(ax + by + c)(dx + ey + f) = 0,$

where a, b, c, d, e, f are real numbers. Clearly the graph of (1) consists of the two lines with equations $ax + by + c = 0$ and $dx + ey + f = 0$. For example, the graph of the equation $x^2 - y^2 = 0$ consists of the two intersecting lines with equations $x + y = 0$ and $x - y = 0$; the graph of $x^2 + 2xy + y^2 = 0$ consists of the two coincident lines with equation $x + y = 0$; the graph of the equation $x^2 + 2xy + y^2 - 1 = 0$ consists of the two parallel lines with equations $x + y + 1 = 0$ and $x + y - 1 = 0$. We shall show that with these three types of exceptions the graph of a second degree polynomial equation in x and y is always a conic section. First we consider the case that $B = 0$.

Theorem 1: Let

$$(2) \qquad Ax^2 + Cy^2 + Dx + Ey + F = 0$$

be a polynomial equation of the second degree which is not factorable and whose graph consists of more than one point. Then the graph of (2) is

(a) an ellipse if $AC > 0$;
(b) a parabola if $AC = 0$;
(c) a hyperbola if $AC < 0$.

Proof: We observe that A and C cannot both be zero since (2) is a second degree equation. Suppose first that $AC \neq 0$, i.e., that $A \neq 0$ and $C \neq 0$. We may complete the square in the terms involving x and in the terms involving y; then (2) can be written in the form

$$(3) \qquad A(x - h)^2 + C(y - k)^2 = M,$$

where

$$h = -\frac{D}{2A}, \quad k = -\frac{E}{2C}, \quad M = \frac{D^2}{4A} + \frac{E^2}{4C} - F.$$

Now $M \neq 0$. For assume the contrary. Then if A and C have the same sign, the graph of (3) and hence also of (2) would consist of the single point (h, k), contrary to our hypothesis. On the other hand, if A and C have unlike signs then the left member of (3) and hence also that of (2) could be factored since it is the difference of two squares, which again contradicts our hypothesis. Hence $M \neq 0$, so (3) may be written in the form

$$(4) \qquad \frac{(x - h)^2}{\frac{M}{A}} + \frac{(y - k)^2}{\frac{M}{C}} = 1.$$

If $AC > 0$, then A and C have the same sign. Thus M must also have this same sign since otherwise (4), and hence (2) would have no graph (for if M had a sign unlike that of A and C, then the left member of (4) would be negative or zero, while the right member is +1). Hence $\frac{M}{A}$ and $\frac{M}{C}$ are both positive, so (4) is the equation of an ellipse with center at (h, k) and with major axis of length $2\sqrt{\frac{M}{A}}$ or $2\sqrt{\frac{M}{C}}$, depending upon the relative magnitudes of $\frac{M}{A}$ and $\frac{M}{C}$. If $AC < 0$ then A and C have opposite signs so (4) is the equation of a hyperbola with center at (h, k).

Suppose now that $AC = 0$; then either $A = 0$ and $C \neq 0$ or $A \neq 0$ and $C = 0$. In the first case, we can complete the square in y in equation (2), writing (2) in the form

$$(5) \quad C(y - k)^2 = -Dx + G,$$

where

$$k = -\frac{E}{2C}, \; G = \frac{E^2}{4C} - F.$$

Now $D \neq 0$; for if $D = 0$ then (5) could be written as

$$(y - k)^2 - \frac{G}{C} = 0,$$

and clearly this equation has no graph if G and C have opposite signs (since in this case $-\frac{G}{C}$ is positive), while the equation is factorable if G and C have the same sign or if $G = 0$. In view of our hypothesis we therefore conclude that $D \neq 0$. Hence (5) may be written in the form

$$(y - k)^2 = -\frac{D}{C}(x - \frac{G}{D}),$$

which is an equation of a parabola with vertex at $(\frac{G}{D}, k)$ and axis parallel to the x-axis.

If $A \neq 0$ and $C = 0$ we may complete the square in x and write (2) in the form

$$(6) \quad A(x - h)^2 = -Ey + H$$

where

$$h = -\frac{D}{2A}, \; H = \frac{D^2}{4A} - F.$$

As above, we conclude that $E \neq 0$ and hence that (6) may be written in the form

$$(x - h)^2 = -\frac{E}{A}(y - \frac{H}{E}),$$

whose graph is a parabola with vertex at $(h, \frac{H}{E})$ and axis parallel to the y-axis. This completes the proof of the theorem.

We shall consider now the general case of a polynomial equation of the second degree in which the coefficient of the xy term is not necessarily zero.

Theorem 2: Let

(7) $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

be a polynomial equation of the second degree in x and y which is not factorable and whose graph consists of more than one point. Then its graph is

(a) an ellipse if $B^2 - 4AC < 0$;
(b) a parabola if $B^2 - 4AC = 0$;
(c) a hyperbola if $B^2 - 4AC > 0$.

Proof: Under a rotation of axes through an arbitrary angle θ, equation (7) is transformed into

(8) $A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0,$

in which the coefficients are given by the equations on page 209.

The graph of (7) consists of more than one point and therefore the same is true for the graph of (8). If (8) were factorable, say as

$(ax' + by' + c)(dx' + ey' + f) = 0,$

then, on replacing x' and y' in this factored form by their values obtainable from (4) on page 208, we would obtain a factorization of (7), contrary to hypothesis. Thus we conclude that for no angle α will equation (8) be factorable.

We now choose the angle of rotation in such a way that $B' = 0$ in (8). [See (5) or (6) in Section 5 of this Lesson.] Under this rotation of axes equation (8) becomes

(9) $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0,$

and we know that this is a second degree equation which is not factorable and whose graph consists of more than one point. But, as we noted on page 94

$B'^2 - 4A'C' = B^2 - 4AC.$

Thus, since $B' = 0$, we have

(10) $-4A'C' = B^2 - 4AC.$

We may now complete our proof by applying the previous theorem to equation (9). If $B^2 - 4AC < 0$ then from (10) $-4A'C' < 0$ and hence $A'C' > 0$. Therefore the graph of (9) and hence also that of (7) is an ellipse. If $B^2 - 4AC = 0$ then, by (10) $-4A'C' = 0$ so $A'C' = 0$. Thus the graph of (9) and of (7) is a parabola. Finally if $B^2 - 4AC > 0$ then, by (10) $-4A'C' > 0$ so $A'C' < 0$ and hence the graph of (9) and of (7) is a hyperbola. This concludes the proof of the theorem.

As a matter of interest we state without proof the following result.

Theorem 3: The graph of the equation

$$(11) \quad Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

if it exists, will be a conic section if the determinant

$$\Delta = \begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix} \neq 0.$$

If $\Delta = 0$ and if the graph of (11) exists then it will be a degenerate conic section; more specifically, the graph of (11) will then be

(a) a single point if $B^2 - 4AC < 0$;
(b) a pair of parallel or coincident lines if $B^2 - 4AC = 0$;
(c) two intersecting lines if $B^2 - 4AC > 0$.

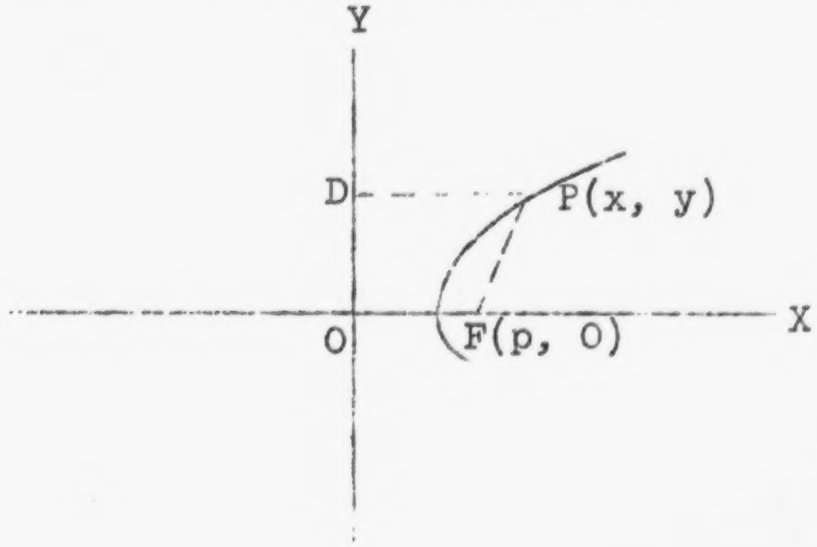
7. Theorem 2 allows us to identify the graph of a second degree polynomial equation provided the graph consists of more than one point and the given equation is not factorable. There is another method which can be used to identify a conic section. We recall that a parabola is bounded in a half-plane but in no proper sector of a half-plane; an ellipse is bounded by a rectangle; and a hyperbola is bounded by two sectors of the plane determined by its asymptotes. Hence for a given second degree polynomial equation we may first determine if the graph of the equation has any (horizontal, vertical, or inclined) asymptotes; if this is the case the graph will be a hyperbola. If the graph has no asymptotes it will be a parabola or an ellipse. If the graph is bounded by a rectangular region it may be an ellipse; if not, it is a parabola.

8. We give here an alternative definition of a conic section (see also the end of Sections 8 and 9 of Chapter 4 in your text). A conic (or conic section) is the locus of all points the ratio of whose distance from a fixed point F to its distance from a fixed line L is a non-negative constant e. This constant is called the eccentricity of the conic. The fixed point is called a focus of the conic and the fixed line is called a directrix of the conic.

Theorem: A conic with eccentricity e is

(a) an ellipse if $e < 1$;
(b) a parabola if $e = 1$;
(c) a hyperbola if $e > 1$.

Proof: Let the conic have focus F and directrix L. Choose a set of coordinate axes so that L coincides with the y-axis and so that F has coordinates (p, 0).



Let P(x, y) be a point of the plane, and let D by the projection of P on the y-axis; then D has coordinates (0, y). By definition, P lies on the conic if and only if

$$\frac{|FP|}{|DP|} = e$$

or

$$|FP| = e\,|DP|.$$

But $|FP| = \sqrt{(x - p)^2 + y^2}$ and $|DP| = \sqrt{x^2}$. Hence P(x, y) is on the conic if and only if

$$\sqrt{(x - p)^2 + y^2} = e\sqrt{x^2}.$$

This equation is equivalent to

$$(x - p)^2 + y^2 = e^2x^2,$$

or

$$(1)\ (1 - e^2)x^2 + y^2 - 2px + p^2 = 0.$$

Clearly equation (1) is of the second degree, and hence its graph is a conic section. Applying Theorem 1 of Section 6 above we see that (1) represents an ellipse if $(1 - e^2) \cdot 1 > 0$, i.e., if $1 - e^2 > 0$ or, since $e > 0$, if $e < 1$. Further, if $(1 - e^2) \cdot 1 = 0$, i.e., if $e = 1$, the graph of (1) is a parabola. Finally if $(1 - e^2) \cdot 1 < 0$, i.e., if $e > 1$, then the graph of (1) is a hyperbola. This completes the proof of the theorem.

In case the conic is a parabola ($e = 1$) the definition coincides with that given on page 182; moreover the focus and directrix have the same meaning in each definition.

In the case of the ellipse or hyperbola the focus as used in this definition coincides with one of the two foci of the previous definitions. Because of the symmetry of the ellipse and the hyperbola it is apparent that another point and another line could serve as focus and as directrix, respectively, of the ellipse and of the hyperbola.

9. Final example.

Given the equation $3x^2 + 2\sqrt{3}xy + y^2 + 2x - 2\sqrt{3}y = 0$. Identify the conic it represents and rotate the coordinate axes through a positive acute angle so chosen that the resulting equation will not have an x'y' term. Sketch the curve, showing both sets of axes.

Solution: Since $B^2 - 4AC = 12 - 4 \cdot 3 \cdot 1 = 0$ the equation represents a parabola. From the given equation we have $A = 3$, $B = 2\sqrt{3}$, $C = 1$, $D = 2$, $E = -2\sqrt{3}$ and $F = 0$. To proceed with the solution of this problem we shall need the following results which were derived either in the text or in these notes.

(i) $\cot \alpha = \frac{A - C}{B} + \sqrt{\left(\frac{A - C}{B}\right)^2 + 1}$

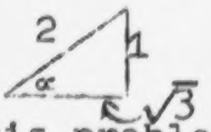
(ii) $x = x' \cos \alpha - y' \sin \alpha$
$y = x' \sin \alpha + y' \cos \alpha$

(iii) $F' = F$

(iv) $A' + C' = A + C$
$A' - C' = B\sqrt{\left(\frac{A - C}{B}\right)^2 + 1}$

Now for our problem we have from (i)

$$\cot \alpha = \frac{3 - 1}{2\sqrt{3}} + \sqrt{\left(\frac{3 - 1}{2\sqrt{3}}\right)^2 + 1} = \sqrt{3}$$

and thus we see from the 

that, $\cos \alpha = \sqrt{\frac{3}{2}}$ and $\sin \alpha = \frac{1}{2}$. (Actually in this problem $\alpha = 30^\circ$.) Hence our equations (ii) become

(v) $x = \sqrt{\frac{3}{2}}x' - \frac{1}{2}y'$

$y = \frac{1}{2}x' + \sqrt{\frac{3}{2}}y'$

We next note from our problem and (iii) that $\boxed{F' = 0}$. Assuming that we have rotated our axes through 30° we then know $\boxed{B' = 0}$ and may use the equations in (iv) to determine A' and C'. We have

$$A' + C' = 3 + 1 = 4$$

$$A' - C' = 2\sqrt{3}\ \sqrt{\left(\frac{3 - 1}{2\sqrt{3}}\right)^2 + 1} = 4$$

or

$A' + C' = 4$ adding, we obtain $\boxed{A' = 4}$, subtracting
$A' - C' = 4$ $\boxed{C' = 0}$.

We need only determine D' and E' in order to complete our solution. To determine D' and E' it is only necessary to substitute the equations given by (v) into the "x" term and "y" term of our given equation (Why?). Thus using (v), $2x - 2\sqrt{3}y$ becomes

$$2\left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right) - 2\sqrt{3}\left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right)$$

$$= \sqrt{3}x' - y' - \sqrt{3}x' - 3y'$$

$= 0 \cdot x' - 4y'$. Hence $\boxed{D' = 0}$ and $\boxed{E' = -4}$.

Finally then the equation

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$$

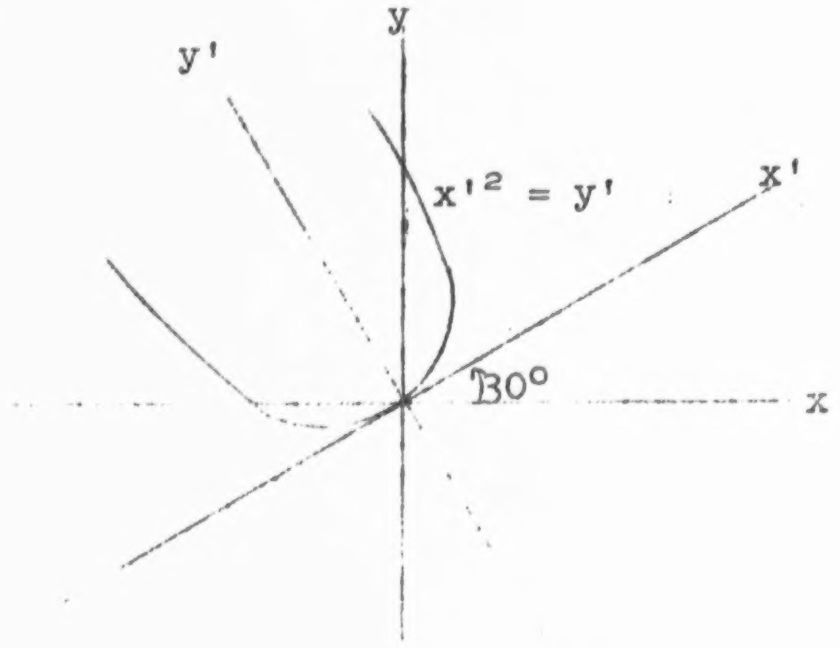
for this particular problem reduces to

$$4x'^2 + 0x'y' + 0y'^2 + 0x' - 4y' + 0 = 0$$

or

$\boxed{x'^2 = y'}$.

Sketch:



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Written Assignment

Solve Problems 1, 4, 5, 8, 9, page 211. Solve Problems 1, 3, 5, 7, 9, pages 213-214.

In addition solve the following problems.

1. Derive the equations of the conics described below. Identify and sketch each conic.

   (a) Focus (1, 2), directrix the line $x + 2y = 1$, eccentricity 1.

   (b) Focus (2, 0), directrix the line $x + 6 = 0$, eccentricity $\frac{1}{3}$.

   (c) Focus (2, -2), directrix the line $x - y + 2 = 0$, eccentricity 2.

2. In each of the following cases, show that the given equation either has no graph or that it represents a degenerate conic:

   (a) $x^2 + y^2 - 4x + 8y + 26 = 0$.

   (b) $x^2 - 6x = y^2 - 6y$.

   (c) $(x + 2y)^2 + 4 = 0$.

3. In each of the following cases, rotate the coordinate axes through a positive acute angle so chosen that the resulting equation will not have an x'y' term. Sketch the curve, showing both sets of axes.

   (a) $16x^2 - 24xy + 9y^2 - 90x - 120y = 0$

   (b) $3x^2 - 10xy + 3y^2 + 32 = 0$

   (c) $5x^2 + 8xy + 5y^2 = 25$

- - - - -

Give the prepared work for Lesson 20 to the supervisor who will complete the mailing procedure.

* * * * *

Note: As a review of Chapter 4 you might wish to solve some of the miscellaneous problems found on pages 217-222 in the text.

* * * * *

When the student has received the evaluated Written Assignment for Lesson 20 and has reviewed the work, he may request the supervisor to administer the hour examination.

* * * * *

Lesson 21

POLAR COORDINATES, GRAPHS, CONICS AND OTHER CURVES

Reference Material

Thomas, Chapter 5, Sections 1, 2, 3

Supplementary Explanation

1. In Section 1 of Chapter 5 we are introduced to a new method for locating a point in the plane, namely, by using polar coordinates. Study this section very carefully. Polar equations of curves will oftentimes be much less complicated than their corresponding rectangular form. Consequently, many results (as we shall see in the next course) are obtained rather easily. To convert an equation from rectangular form to polar form we use the following equations (see (2), page 225):

   (1) $x = r\cos\theta$, $y = r\sin\theta$ and hence $x^2 + y^2 = r^2$.

   We also have $\frac{y}{x} = \tan\theta$.

2. Examples:

   (A) Find an equation whose rectangular graph is the same as the polar graph of $r = \sin 2\theta$.

   Solution: We are given $r = \sin 2\theta$. This may be written as $r = 2\sin\theta\cos\theta$. Now using the equations given in (1) above we have

   $$\sqrt{x^2 + y^2} = 2\frac{y}{\sqrt{x^2 + y^2}} \cdot \frac{x}{\sqrt{x^2 + y^2}} \text{ or}$$

   $(x^2 + y^2)^{\frac{3}{2}} = 2xy$ and squaring both sides this becomes

   $(x^2 + y^2)^3 = 4x^2y^2$.

   (B) Find an equation whose polar graph is the same as the rectangular graph of $(x^2 + y^2)(x - a)^2 = b^2x^2$.

   Solution: Again using the equations in (1) above our given equation becomes

   $r^2(r\cos\theta - a)^2 = b^2r^2\cos^2\theta$.

   This equation implies that either $r = 0$ or $(r\cos\theta - a)^2 = b^2\cos^2\theta$. This reduces to

   $r\cos\theta - a = \pm b\cos\theta$ or

   $$r = \frac{a \pm b\cos\theta}{\cos\theta} = a\sec\theta \pm b.$$

3. To sketch a curve whose equation is given in polar coordinates, a table of values can often be drawn up and a point-by-point plot made, but it is usually more efficient first to analyze the equation for symmetry, extent, excluded values, intercepts, tangents at the pole, etc. (You may also wish to purchase some "polar paper" at your local bookstore.)

(a) Symmetry: It can be readily seen from a figure that the point (r, θ) is symmetric to thepoint $(-r, \theta)$ or to the point $(r, \pi + \theta)$ with respect to the pole; furthermore, the point (r, θ) is symmetric to the point $(r, -\theta)$ or to the point $(-r, \pi - \theta)$ with respect to the polar axis; again, the point (r, θ) is symmetric to the point $(-r, -\theta)$ or to the point $(r, \pi - \theta)$ with respect to the 90° -axis.

These remarks yield the following facts:

Symmetry tests: The graph of an equation in polar coordinates r and θ has:

(i) the pole as a center of symmetry if the equation is unchanged when r is replaced by $-r$, or when θ is replaced by $\pi + \theta$;

(ii) the polar axis as an axis of symmetry if the equation is unchanged when θ is replaced by $-\theta$, or when r is replaced by $-r$ and θ by $\pi - \theta$;

(iii) the 90° - axis as an axis of symmetry if the equation is unchanged when θ is replaced by $\pi - \theta$, or when r is replaced by $-r$ and θ by $-\theta$.

(b) Extent and excluded values: Values of θ for which r becomes imaginary will give excluded regions. The extent of the graph is indicated by finding values, if any, which make r a maximum or minimum, or which make r become infinite, and by consideration of excluded regions, if any.

(c) Intercepts: Intercepts of the graph on the polar axis or its extension through the pole are evidently found by putting $\theta = 0, \pm\pi, \pm 2\pi$, etc., and solving for r. Similarly, intercepts on the 90° -axis or its extension through the pole are obtained by putting $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$, etc., and solving for r.

(d) Tangents at the pole: If $r = f(\theta)$ is continuous at $\theta = \theta_0$ and $f(\theta_0) = 0$ then the graph of $\theta = \theta_0$ is a line tangent to the graph of $r = f(\theta)$ at the pole. Indeed, if the point $Q(\Delta r, \theta_0 + \Delta\theta)$ is a second point on the curve (Figure 1), then the secant line OQ has the equation $\theta = \theta_0 + \Delta\theta$, since any line through the pole has the equation of the form $\theta =$ constant.

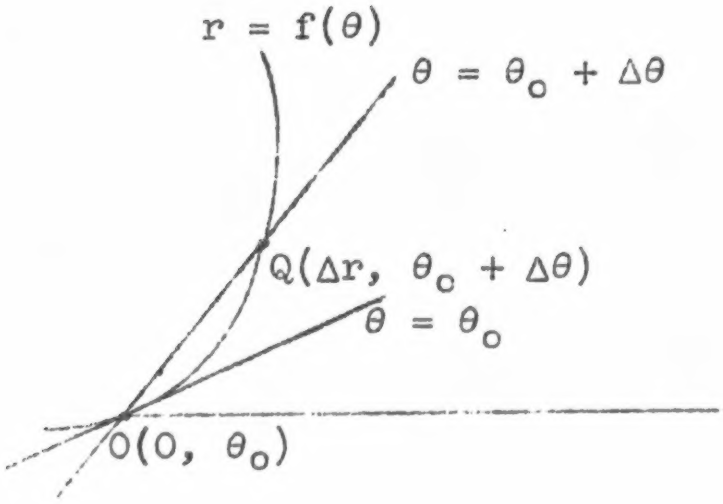

$r = f(\theta)$
$\theta = \theta_o + \Delta\theta$
$Q(\Delta r, \theta_o + \Delta\theta)$
$\theta = \theta_o$
$O(0, \theta_o)$

Figure 1

When $\Delta\theta$ approaches zero, then also Δr approaches zero because by assumption $r = f(\theta)$ is a continuous at $\theta = \theta_o$. Thus, when $\Delta\theta$ approaches zero, Q approaches the pole O along the curve; and the secant line, $\theta = \theta_o + \Delta\theta$, approaches $\theta = \theta_o$, which by definition must be the tangent line to the graph of $r = f(\theta)$ at the pole. Thus if a curve passes through the pole, the directions (and equations) of any tangent lines to the curve at the pole are found by putting $r = 0$ in the equation of the curve and solving for θ.

4. We shall discuss the graph of $r = a + b \cos \theta$, $a > 0$, $b > 0$.

Symmetry: Since $a + b \cos \theta = a + b \cos (-\theta)$, we see from the symmetry tests that the graph of $r = a + b \cos \theta$ is symmetric with respect to the polar axis and its extension.

Extent and excluded values: For each value of θ the corresponding value of r is clearly a real number so that no values are excluded for θ. Since $| \cos \theta | \leq 1$ for each value of θ, $| r | \leq a + b$.

Intercepts: If $\theta = 0$ then $r = a + b$, and if $\theta = \pm\pi$ then $r = a - b$, so that the graph of $r = a + b \cos \theta$ intersects the polar axis or its extension at the points $A(a + b, 0)$ and $B(a - b, \pi)$. If $a \leq b$ then the graph of $r = a + b \cos \theta$ will pass through the pole O since $r = 0$ for any angle θ such that $\cos \theta = -\frac{a}{b}$. For $a > b$ we have, of course, $r > 0$ for every θ. If $\theta = \frac{1}{2}\pi$ or $\theta = \frac{3}{2}\pi$ then $r = a$ so that the graph of $r = a + b \cos \theta$ intersects the 90° -axis and its extension at the points

$C\left(a, \frac{1}{2}\pi\right)$ and $D\left(a, \frac{3}{2}\pi\right)$.

Tangents at the pole: The graph of $r = a + b\cos\theta$ passes through the pole if $a \leq b$. In this case, putting $r = 0$ and solving for θ, we obtain $\theta = \cos^{-1}\left(\frac{-a}{b}\right)$ which yields the equations for the tangent lines at the pole.

In Figure 2 we have sketched the graphs of $r = a + b\cos\theta$ for $a < b$, $a = b$, and $a > b$. If θ varies from 0 to 2π the graphs are traced out by starting with A and continuing as indicated by the arrowheads.

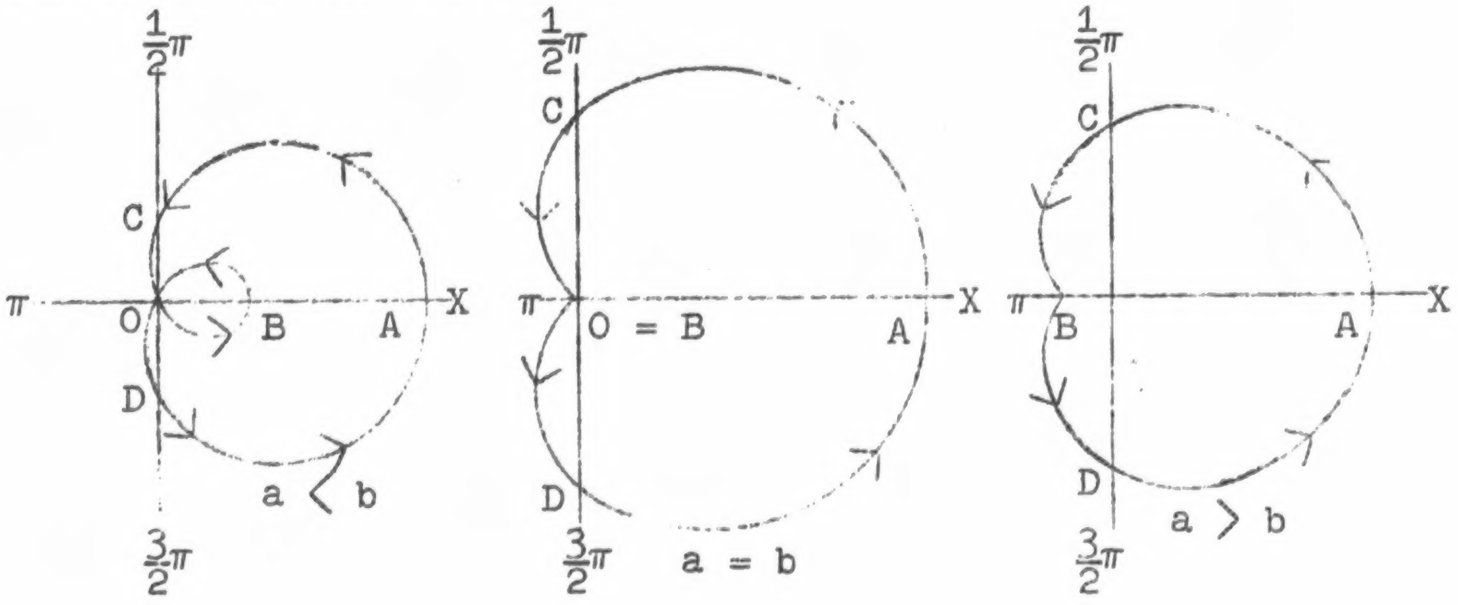


Figure 2

5. Since $a - b\cos\theta = a + b\cos(\theta - \pi)$, we may obtain the graph of $r = a - b\cos\theta$ from the graph of $r = a + b\cos\theta$ by a rotation of 180° about the pole. Again, since $a + b\sin\theta = a + b\cos\left(\theta - \frac{\pi}{2}\right)$, the graph of $r = a + b\sin\theta$ may be obtained from that of $r = a + b\cos\theta$ by a counterclockwise rotation of 90° about the pole. Analogously, since $a - b\sin\theta = a + b\cos\left(\theta - \frac{3}{2}\pi\right)$, the graph of $r = a - b\sin\theta$ can be obtained from the graph of $r = a + b\cos\theta$ by a counterclockwise rotation of 270° about the pole.

   Each of the graphs of $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$ is called a limaçon (which means "snail"). If $a = b$ the limaçon is also called a cardioid (meaning "heart shaped"). If $a = \frac{1}{2}b$ then the curve is called the trisectrix. [See suggestion 7 which follows for its significance.]

   The rectangular-coordinate equation of $r = a \pm b\cos\theta$ can be found to be $(x^2 + y^2 \mp bx)^2 = a^2(x^2 + y^2)$. The rectangular-coordinate equation of $r = a \pm b\sin\theta$ is $(x^2 + y^2 \mp by)^2 = a^2(x^2 + y^2)$.

6. A limaçon may be described as the locus of a point P on a line L, at a fixed distance from the intersection of L with a fixed circle C, as L revolves about a fixed point O on the circle. Let the fixed distance be taken as a, the diameter of the fixed circle C as b, the fixed point O as the pole, and the diameter of C on the polar axis (Figure 3).

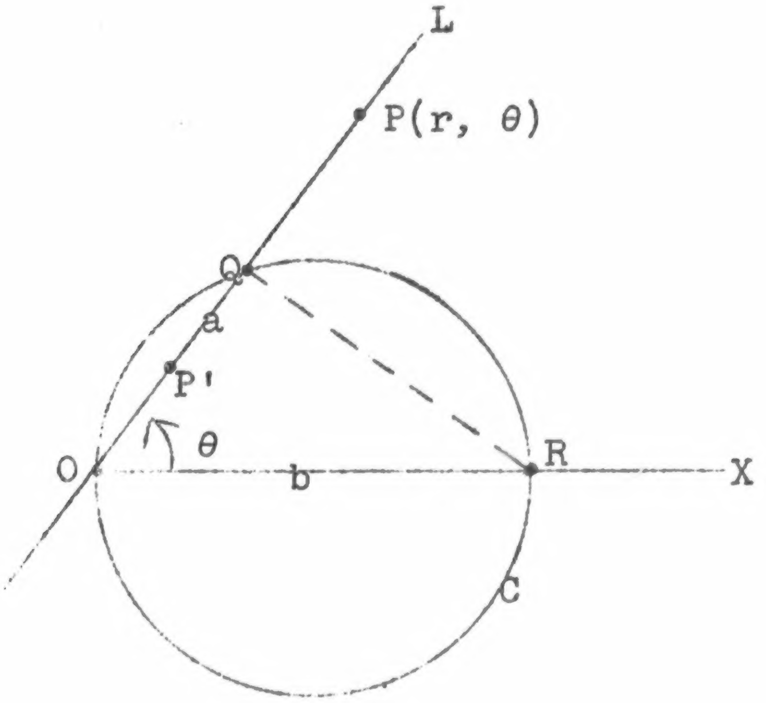


Figure 3

From the right triangle OQR we obtain at once $\overrightarrow{OQ} = b \cos \theta$. Since $\overline{QP} = a$ and $\overline{OP} = r$ we obtain therefore $r = \overline{OP} = \overline{OQ} + \overline{QP} = b \cos \theta \pm a$. Since the graphs of $r = a + b \cos \theta$ and $r = -a + b \cos \theta = -a + b \cos (-\theta)$ are the same, we have $r = a + b \cos \theta$ as an equation for the limaçon.

7. Let us find the locus of the vertex P of a triangle OPA whose side OA is the length a and for which angle OPA is one-half of angle POA.

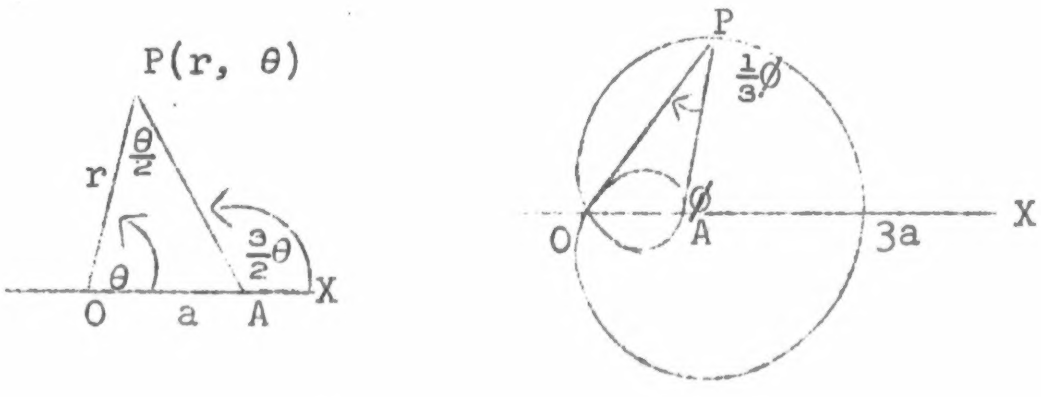


Figure 4

In Figure 4 we have chosen the vertex O as the pole of a polar coordinate system and the side OA along the polar axis. Since the sum of the angles of a triangle is π, angle OAP must be $\pi - \frac{3}{2}\theta$.

By the law of sines we have

$$\frac{r}{\sin\left(\pi - \frac{3}{2}\theta\right)} = \frac{a}{\sin\frac{\theta}{2}}.$$

Hence

$$r = a \cdot \frac{\sin\left(\pi - \frac{3}{2}\theta\right)}{\sin\frac{\theta}{2}} = a \cdot \frac{\sin\frac{3}{2}\theta}{\sin\frac{\theta}{2}}.$$

Since $\sin\frac{3}{2}\theta = 3\sin\frac{\theta}{2} - 4\sin^3\frac{\theta}{2}$ and $2\sin^2\frac{\theta}{2} = 1 - \cos\theta$, we obtain

$$\begin{aligned} r &= a \cdot \frac{3\sin\frac{\theta}{2} - 4\sin^3\frac{\theta}{2}}{\sin\frac{\theta}{2}} \\ &= a\left(3 - 4\sin^2\frac{\theta}{2}\right) \\ &= a(3 - 2(1 - \cos\theta)). \end{aligned}$$

Thus

$$r = a + 2a\cos\theta.$$

From this equation we see that the locus of P is a limaçon (with $b = 2a$). This limaçon may be used to trisect an angle. Angle OPA is one-third the exterior angle PAX. Hence, to trisect an angle ϕ, $0 < \phi < \pi$, we place its vertex at A and its initial side along the polar axis. Let its terminal side intersect the (bigger loop of the) limaçon $r = a + 2a \cos\theta$ at P. Draw OP. Then angle APO will be $\frac{1}{3}\phi$.

8. We next prove that the distance between the points $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$ is given by $|PQ|^2 = r_1{}^2 + r_2{}^2 - 2r_1r_2 \cos(\theta_2 - \theta_1)$. (See also Example 3, page 233). Let $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$ be any given points and let θ be the measure of angle POQ (Figure 5) with $0 \leq \theta \leq 180^\circ$. By the law of cosines we have

$$|PQ|^2 = |OP|^2 + |OQ|^2 - 2 \cdot |OP| \cdot |OQ| \cdot \cos\theta$$

or, since $|OP| = |r_1|$ and $|OQ| = |r_2|$,

(1) $$|PQ|^2 = r_1^2 + r_2^2 - 2 \cdot |r_1| \cdot |r_2| \cdot \cos\theta.$$

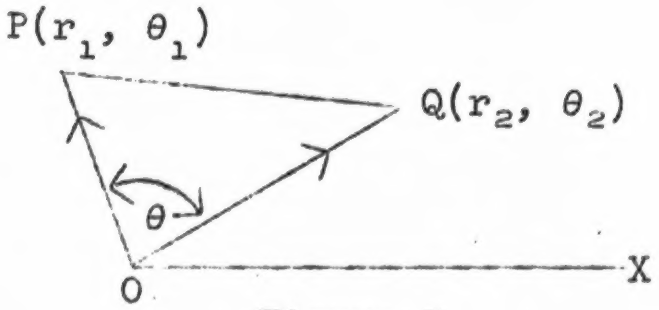


Figure 5

If r_1 and r_2 have the same sign the $|r_1| \cdot |r_2| = r_1 \cdot r_2$ and obviously $\theta = |\theta_2 - \theta_1| \pm 2n\pi$ where n is some integer. Thus $\cos\theta = \cos[|\theta_2 - \theta_1| \pm 2n\pi] = \cos(\theta_2 - \theta_1)$, and (1) becomes

$$(2) \qquad |PQ|^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1).$$

If r_1 and r_2 have opposite signs then $|r_1| \cdot |r_2| = -r_1 \cdot r_2$. But in this case we have $\theta = \pi - |\theta_2 - \theta_1| \pm 2n\pi$, where n is some integer, and thus $\cos\theta = \cos[\pi - |\theta_2 - \theta_1| \pm 2n\pi] = -\cos(\theta_2 - \theta_1)$. Hence also in this case we obtain (2). Thus (2) holds in any case.

9. We next prove that the equation

$$(1) \qquad r = \frac{2ep}{1 - e\cos\theta}$$

is the equation of a hyperbola if $e > 1$. Note that this equation is the same as that given by Equation (13) on page 234 if we let $k = 2p > 0$. Thus p represents the distance from a focus to a vertex.

Equation (1) may be written in the form

$$(2) \qquad r = e(r\cos\theta + 2p).$$

From (1) we see that $r > 0$ for $1 - e\cos\theta > 0$ and $r < 0$ for $1 - e\cos\theta < 0$. By the equations (1) of Section 1 of these notes we have therefore that the graph of (2) has equations

$$(3) \qquad \pm\sqrt{x^2 + y^2} = e(x + 2p)$$

in rectangular coordinates. Again, the graph of (3) can be shown to be the same as the graph of

$$(4) \qquad x^2 + y^2 = e^2(x^2 + 4px + 4p^2).$$

If $e > 1$, we may complete squares and put (4) in the form

$$(5) \qquad \left(x + \frac{2e^2p}{e^2 - 1}\right)^2 \quad \frac{y^2}{e^2 - 1} = \frac{4e^2p^2}{(e^2 - 1)^2}.$$

Equation (5) is the equation of a hyperbola with foci on the x-axis and center at $\left(-\frac{2e^2p}{e^2 - 1}, 0\right)$.

Furthermore

$$a^2 = \frac{4e^2p^2}{(e^2 - 1)^2}, \quad b^2 = \frac{4e^2p^2}{e^2 - 1}$$

and hence

$$c^2 = a^2 + b^2 = \frac{4e^4p^2}{(e^2 - 1)^2}.$$

Since $c = \frac{2e^2p}{e^2 - 1}$ and the center is at $\left(-\frac{2e^2p}{e^2 - 1}, 0\right)$, we see that one focus is at the origin 0 while the other focus has coordinates $\left(-\frac{4e^2p}{e^2 - 1}, 0\right)$.

10. We note that equation (13) namely $r = \frac{ke}{1 - e\cos\theta}$ on page 235 represents various conics (the type depending on the value of e) each of which has a directrix to the left of the pole. In like manner one may derive the equations of the following conics (again, the type depending on the value of e) each of which will have a focus at the pole.

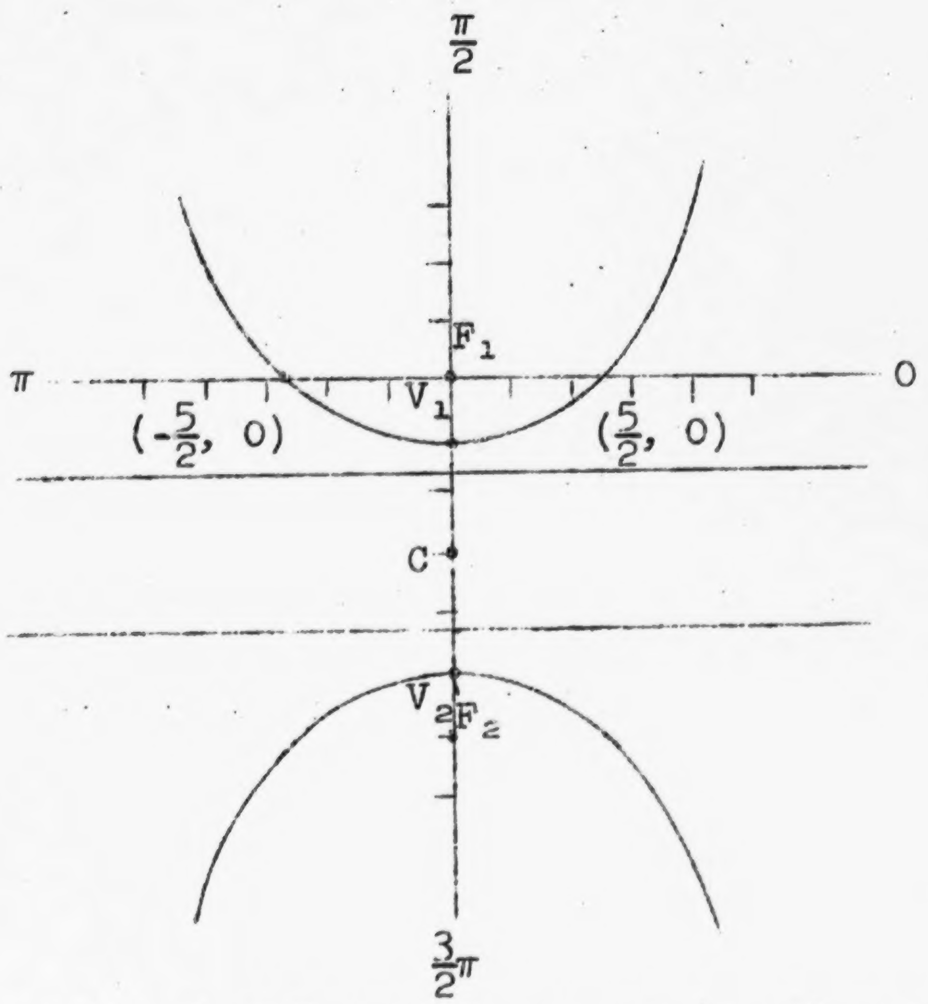
$r = \frac{ke}{1 + e\cos\theta}$	[Directrix k units to the right of pole.]
$r = \frac{ke}{1 + e\sin\theta}$	[Directrix parallel to the polar axis (x-axis) and k units above it.]
$r = \frac{ke}{1 - e\sin\theta}$	[Directrix parallel to the polar axis and k units below it.]

11. Example: Describe and sketch $r = \frac{5}{2 - 3\sin\theta}$.

Sketch: We compare this equation with $r = \frac{ke}{1 - e\sin\theta}$ given in Section 10 above. Now $r = \frac{5}{2 - 3\sin\theta} = \frac{\frac{5}{2}}{1 - \frac{3}{2}\sin\theta}$.

Hence $e = \frac{3}{2}$ and consequently the conic is a hyperbola. Since $e = \frac{3}{2}$ and $ke = \frac{5}{2}$ we have $k = \frac{5}{3}$. Thus our hyperbola has a directrix $\frac{5}{3}$ units below the polar axis. We note that it is symmetric with respect to $\theta = \frac{\pi}{2}$ and further when

$\theta = 0$, $r = \pm\frac{5}{2}$; when $\theta = \frac{\pi}{2}$, $r = -5$, when $\theta = \frac{3}{2}\pi$, $r = 1$. Plotting these results we have the following sketch.



We note that the Foci F_1 and F_2 are at $(0, 0)$ and $(-6, \frac{\pi}{2})$ and the Verticies V_1 and V_2 are at $(-1, \frac{\pi}{2})$ and $(+5, \frac{3}{2}\pi)$ and the Center C is at $(-3, \frac{\pi}{2})$ and the directrix corresponding to the focus at the pole will have rectangular equation $y = -\frac{5}{3}$ or polar equation $r \sin \theta = -\frac{5}{3}$.

12. Note that on page 235 of the text the author asserts that "ke is replaced by its equivalent value"

$$ke = a(1 - e^2).$$

This result follows from the fact that k represents the distance from the focus to the directrix. Hence from Figure 4-39, page 195 it follows that

$$k = \frac{a}{e} - c \text{ or}$$

$$k = \frac{a - ec}{e}, \text{ but } e = \frac{c}{a} \text{ (see (12) on page 195 of the text)}$$

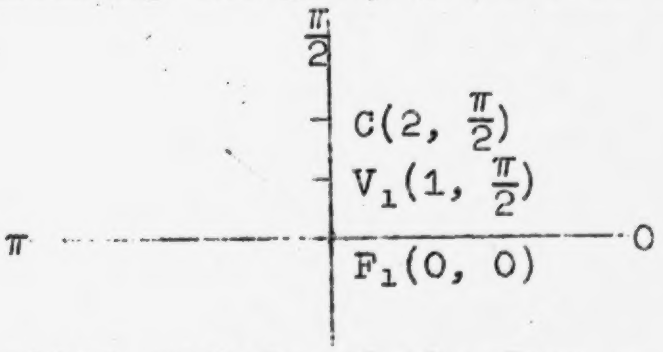
consequently

$$k = \frac{a - e(ea)}{e} = \frac{a(1 - e^2)}{e}$$

and thus $ke = a(1 - e^2)$.

13. Problem 26, page 237 of the text

Solution: Sketching the given points we have



Thus the hyperbola must be of the form $r = \frac{ke}{1 + e \sin \theta}$. We must find e and ke. Referring to the given points and Figure 4-49, page 205 of the text we see that $a = 1$. Further since k represents the distance from the focus to a directrix we have

$$k = c - \frac{a}{e} = \frac{ce - a}{e} = \frac{ae^2 - a}{e}, \text{ since } e = \frac{c}{a}.$$

Thus $ke = a(e^2 - 1)$. Now ae represents the distance from the center to the focus. Thus for our problems $ae = 2$ hence $e = 2$ since $a = 1$. Consequently $ke = 1(4 - 1) = 3$. Our desired equation is given by

$$r = \frac{3}{1 + 2 \sin \theta}.$$

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Written Assignment

Solve Problems 1a, b, 2aeg, 3, 4ab, 5, 7, pages 226-227. Solve Problems 1, 3, 5, 6, page 230. Solve Problems 3, 7, 11, 12, 17, 20, pages 235-236. Solve Problems 13b, 19, 25, page 237.

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Give the prepared work for Lesson 21 to the supervisor who will complete the mailing procedure.

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Note: The student should at this time make a systematic review of all the work assigned in this course. He may review all the work completed for the Written Assignments, the Hour Examinations, and textbook materials.

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When all of the Written Assignments for this course have been returned to the student, he may request the supervisor to administer the Final Examination.

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This completes the work of this course.

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